11.5 – 11.6

Alternating Series

Convergence tests work for series of positive terms.
What about **mixed** terms?
Alternating series: alternately positive and negative.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

$$= \sum (-1)^{n-1} \frac{1}{n}$$

Alternating series test:– (~~Lieb~~ Leibnitz' test)

If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$, $a_n > 0$ satisfies

1) $a_n \geq a_{n+1}$ for all n (decreasing)

11) $\lim_{n\to\infty} a_n = 0$

then, the series is convergent.

Exm: $\sum (-1)^{n-1} \frac{1}{n}$.

$a_n = \frac{1}{n} \geq \frac{1}{n+1} = a_{n+1}$

and $a_n = \frac{1}{n} \to 0$ as $n \to \infty$

So the series is convergent.

Ex 11.7

7. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$

$\displaystyle\lim_{n\to\infty} \frac{3n-1}{2n+1} = 3/2 \neq 0.$ So no alternating test!

Also, $\displaystyle\lim_{n\to\infty} \frac{(-1)^n(3n-1)}{2n+1}$ does not exist. So divergent.

11. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}.$

$\displaystyle\lim_{n\to\infty} \frac{n^2}{n^3+4} = 0.$

To show, $a_n = \dfrac{n^2}{n^3+4}$ is decreasing.

$f(x) = \dfrac{x^2}{x^3+4}$

$f'(x) = \dfrac{8x - x^4}{(x^3+4)^2} < 0$ as $8 - x^3 < 0$

$\Rightarrow x > 2$

So, $a_n \geq a_{n+1}$ for $n \geq 3$.

Also, $a_1 = \frac{1}{2} > \frac{4}{9} = a_2$

So, $\{a_n\}$ is decreasing. Hence, convergent.

15. $\displaystyle\sum_{n=0}^{\infty} \frac{\sin(n+1/2)\pi}{1+\sqrt{n}}$

$\sin(n\pi + \pi/2) = (-1)^n \sin \pi/2 = (-1)^n.$

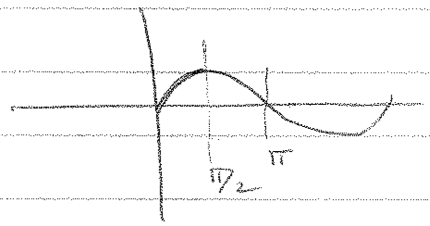
$= \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}.$ ∴ Convergent.

(3)

17. $\sum_{n=1}^{\infty} (-1)^n \sin\frac{\pi}{n} = \sum_{n=1}^{\infty} (-1)^n a_n = a_2 - a_3 + a_4 - a_5 + \cdots$

$a_1 = \sin\pi = 0$

$a_2 = \sin\pi/2 = 1$, $a_3 = \sin\pi/3 = \sqrt{3}/2$



~~Since as~~

$f(x) = \sin(\pi/x)$ is decreasing in $[2, \infty)$.

or

$f'(x) = -\cos(\pi/x) \cdot \pi/x^2 < 0$

if $\cos(\pi/x) > 0$

i.e. $2 < x < \infty$

Also, $\lim_{n\to\infty} \sin(\pi/n) = \cancel{\sin(0)}$

$= \sin\left(\lim_{n\to\infty} \pi/n\right)$ [$\sin x$ is a continuous fn]

$= \sin 0 = 0.$

So, $\sum (-1)^n a_n$ is convergent.

11.6 Absolute convergence

Defn: $\sum a_n$ is absolutely convergent if $\sum |a_n|$ is convergent.

$\sum a_n$ is a series of +ve terms, i.e. $a_n \geq 0 \ \forall n$. then,

$$\sum |a_n| = \sum a_n.$$

Exm:– $\sum (-1)^{n-1} \frac{1}{n^3}$ is absolutely convergent as $\sum \frac{1}{n^3}$ is convergent. (p-series).

Defn:– $\sum a_n$ is conditionally convergent if it is convergent but not absolutely convergent.

Exm:– $\sum (-1)^{n-1} \frac{1}{n}$.

Theorem:– Absolutely convergent $\Rightarrow$ convergent.

Pf:– $0 \leq a_n + |a_n| \leq 2|a_n|$

Now, $2\sum |a_n|$ is convergent, so by comparison test, $\sum (a_n + |a_n|)$ is convergent.

Now, $\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$

Both series of RHS are convergent, so is $\sum a_n$.

Exm:– $\sum \frac{\sin n}{n^3}$ $\quad \frac{|\sin n|}{n^3} \leq \frac{1}{n^3}$.

So, $\sum \frac{\sin n}{n^3}$ is absolutely convergent, so convergent.

Ratio Test :-

i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, $\sum a_n$ is absolutely conv.

ii) if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or ∞, then, $\sum a_n$ is divergent.

iii) if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the test is inconclusive.

Exm :- (iii) $\sum \frac{1}{n^2}$, $\sum \frac{1}{n}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{n^2} \to 1 \text{ as } n\to\infty, \text{ conv.}$$

$$\frac{b_{n+1}}{b_n} = \frac{n+1}{n} \to 1 \text{ as } n\to\infty, \text{ div.}$$

Exm :- $\sum \frac{n^n}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{(n+1)!}\,\frac{n!}{n^n}$$

$$= \frac{(n+1)\,(n+1)^n\;n!}{(n+1)\,n!\,.\,n^n} = \left(1+\frac{1}{n}\right)^n \to e \text{ as } n\to\infty$$

As $e > 1$, the series is divergent by ratio test.

Or $\frac{n^n}{n!} = \frac{n.n.n...n}{1.2.3.4..n} \geq n$. So, $a_n \not\to 0$.

Root test :-

Let, $\lim_{n\to\infty} (|a_n|)^{1/n} = L$.

i) If $L < 1$, $\sum a_n$ is absolutely conv.

ii) If $L > 1$ or $L = \infty$, the $\sum a_n$ is div.

iii) if $L = 1$, the test is inconclusive.

(If $L = 1$, don't try ratio test, it will be inconclusive too)

Rearrangement :-

$$a_1 + a_2 + a_3 + a_4 + a_5 = a_1 + a_2 + a_5 + a_3 + a_4 \quad ✓$$

But, $a_1 + a_2 + \cdots \infty \neq a_1 + a_2 + a_5 + a_3 + a_4 + a_{10} + a_6 + \cdots \infty$

Result:- If $\sum a_n$ is absolutely conv. with sum S, then, any rearrangement of $\sum a_n$ has sum S.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln 2 \quad \cdots (I)$$

$(I) \times \frac{1}{2}$

$$0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + 0 \cdots = \frac{1}{2}\ln 2 \quad \cdot (II)$$

$(I) + (II)$

$$\Rightarrow 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} \cdots = \frac{3}{2}\ln 2 . \quad \cdots (III)$$

(III) is a rearrangement of (I).

If Σa_n is conditionally convergent series and r is ANY real number, then, $\exists$ a rearrangement of Σa_n that has ~~the~~ sum r. (Riemann)

The sum $= \ln\left(2\sqrt{\frac{p}{n}}\right)$ p = no of +ve terms

n = no of −ve terms.

$= \ln 2 + \frac{1}{2}\ln(p/n)$

Exm:- $\displaystyle\sum_{n=1}^{\infty} \frac{\ln 5n}{n^7}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\ln 5(n+1)}{\ln 5n}\,\frac{n^7}{(n+1)^7} \to 1 \text{ as } n \to \infty$$

But, $\ln x \le (x-1)$ for $x \ge 1$

as $\lim_{x\to\infty} \frac{\ln x}{x} = 0$

or $f(x) = x - \ln x$

$f'(x) = 1 - \frac{1}{x} > 0$ for $x > 1$

$f(x) \geq f(1)$

$x - \ln x \geq 1$

$\Rightarrow (x-1) \geq \ln x$

So, $\displaystyle \frac{\ln 5n}{n^7} \le \frac{(5n-1)}{n^7}$

$\displaystyle = \frac{5}{n^6} - \frac{1}{n^7}$

$\Sigma\left(\frac{5}{n^6} - \frac{1}{n^7}\right)$ convergent, so by Comparison test Σa_n is convergent.

$\sum \dfrac{(n+1)(10^2-1)^n}{10^{2n}}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+2}{n+1}\,\frac{(10^2-1)}{10^2} \longrightarrow \frac{10^2-1}{10^2} < 1 \text{ as } n\to\infty$$

So, absolutely convergent.

<u>19</u> $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$

$$\left|\frac{\cos n\pi/3}{n!}\right| \le \frac{1}{n!} \le \frac{1}{n(n-1)} \quad \text{for } n \ge 2$$

$\sum \frac{1}{n(n-1)}$ telescoping series, convergent. So does

$$\sum \left|\frac{\cos n\pi/3}{n!}\right| \quad \& \quad \sum \frac{\cos n\pi/3}{n!}.$$

<u>25</u> $\displaystyle\sum \frac{n^{100}\,100^n}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{100}\,100}{(n+1)\;n^{100}} = \frac{\left(1+\frac{1}{n}\right)^{100}\cdot 100}{(n+1)} \to 0 \text{ as } n\to\infty$$

So, $\displaystyle\sum \frac{n^{100}\,100^n}{n!}$ is absolutely convergent.