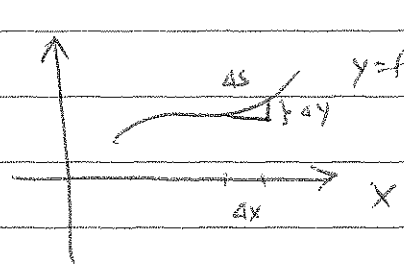
8.1 ①



## Arc Length :-

$y = f(x)$ curve length




$(ds)^2 = (dx)^2 + (dy)^2$ , by Pythagorus thm

$\Rightarrow ds = \sqrt{(dx)^2 + (dy)^2}$

$\therefore ds = \sqrt{dx^2 + dy^2}$

$\therefore S = \int_a^b ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$

$= \int_a^b \sqrt{1 + f'(x)^2}\, dx.$ , provided $f'(x)$ is cont. on $[a,b]$.

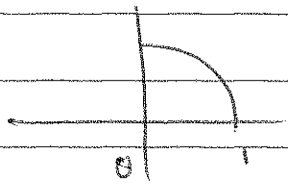
Also, $S = \int_c^d \sqrt{1 + g'(y)^2}\, dy$

Exm :- Find the length of the first quarter of the circle $y = \sqrt{1-x^2}$

$S = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$y = \sqrt{1-x^2}$

$\frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}}$




$= \int_0^1 \sqrt{1 + \frac{x^2}{1-x^2}}\, dx$

$= \int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x \Big]_0^1 = \pi/2.$

(Improper Integral) at 1

<u>EXM</u> :- $y = \left(\frac{x}{2}\right)^{2/3}$ , $x = 0$ to $x = 2$ .

$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{3}\right)^{-1/3}$ discontinuous at $x = 0$

$y^{3/2} = x/2 \Rightarrow x = 2y^{3/2}$

$\therefore \frac{dx}{dy} = 3\sqrt{y}$.

$x = 0 \Rightarrow y = 0$

$x = 2 \Rightarrow y = 1$

So, $S = \int_0^1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + 9y}\, dy$ $\qquad 1 + 9y = u$

$9\,dy = du$

$= \frac{1}{9}\int_1^{10} \sqrt{u}\, du$

$= \frac{1}{9}\,\frac{2}{3}\, u^{3/2}\Big]_1^{10} = \frac{2}{27}\left(10^{3/2} - 1\right)$

<u>Arc length function</u> :-

$$S(x) = \int_a^x \sqrt{1 + f'(t)^2}\, dt$$

<u>35</u> $\quad y = \sin^{-1}x + \sqrt{1 - x^2} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} - \frac{x}{\sqrt{1 - x^2}}$

$\therefore S(x) = \int_0^x \sqrt{1 + \frac{(1-t)^2}{(1-t^2)}}\, dt$

$= \int_0^x \sqrt{1 + \frac{1-t}{1+t}}\, dt = \sqrt{2}\int_0^x (1+t)^{-1/2}\, dt$

$= 2\sqrt{2}\,(1+t)^{1/2}\Big]_0^x$

$= 2\sqrt{2}\left(\sqrt{1+x} - 1\right)$

③

<u>Exm</u> $x = 2 + \frac{1}{4}\cosh(4y)$ , $y=0$ to $y=6$.

$$\frac{dx}{dy} = \sinh(4y)$$

$$\therefore S = \int_0^6 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$= \int_0^6 \sqrt{1+\sinh^2 4y}\,dy$$

$$= \int_0^6 \cosh(4y)\,dy = \left.\frac{\sinh(4y)}{4}\right]_0^6 = \frac{\sinh(24)}{4}.$$

<u>Exm</u>. Find $y=f(x)$ through $(16,1)$ with length

$$L = \int_{16}^{21} \sqrt{1+\frac{1}{64x}}\,dx. \quad \text{on } [16,21]$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{64x} \Rightarrow \frac{dy}{dx} = \frac{1}{8\sqrt{x}}.$$

$$\therefore y = \int \frac{1}{8\sqrt{x}}\,dx$$

$$= \frac{1}{8}\cdot 2\sqrt{x} + c.$$

$$y(x) = \frac{1}{4}\sqrt{x} + c.$$

$$y(16) = 1 \Rightarrow 1 = \frac{1}{4}4. + c \Rightarrow c = 0$$

$$\text{So, } f(x) = \frac{\sqrt{x}}{4}.$$

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$$y = \ln(\sec x) \qquad 0 \le x \le \pi/4$$

$$\frac{dy}{dx} = \frac{1}{\sec x} \sec x \tan x$$

$$S = \int_0^{\pi/4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx = \int_0^{\pi/4} \sec x \, dx$$

$$= \ln|\sec x + \tan x| \Big]_0^{\pi/4}$$

$$= \ln(\sqrt{2} + 1)$$

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$$y = \ln(1 - x^2)$$

$$\frac{dy}{dx} = \frac{-2x}{1 - x^2} \qquad S = \int_0^{1/2} \sqrt{1 + \frac{4x^2}{(1-x^2)^2}}\, dx$$

$$= \int_0^{1/2} \sqrt{\frac{(1+x^2)^2}{(1-x^2)^2}}\, dx$$

$$= \int_0^{1/2} \frac{(1+x^2)}{(1-x^2)}\, dx$$

$$= -\int_0^{1/2} dx + 2\int_0^{1/2} \frac{dx}{(1-x)(1+x)}$$

$$= -\frac{1}{2} + \int_0^{1/2} \left[\frac{1}{1-x} + \frac{1}{1+x}\right] dx$$

$$= -\frac{1}{2} + \Big[\ln|1+x| - \ln|1-x|\Big]_0^{1/2}$$

$$= -\frac{1}{2} + \ln\left(\frac{3}{2}\right) - \ln\left(\frac{1}{2}\right)$$

$$= \underline{\underline{\ln(3) - \frac{1}{2}}}$$