11.4 Comparison test:

Test I: Suppose $\sum a_n$ & $\sum b_n$ are series of positive terms. and $a_n \le b_n$ for all n.

i) if $\sum b_n$ is convergent, $\sum a_n$ is convergent.

ii) if $\sum a_n$ is divergent, $\sum b_n$ is divergent.

Test II: Suppose $\sum a_n$ & $\sum b_n$ are series of positive terms. and

$\lim_{n\to\infty} \frac{a_n}{b_n} = c$, a finite number.

Then, either both series converge or diverge.

Exm:- $\sum \frac{1}{n^{10}+4}$

$$\frac{1}{n^{10}+4} \le \frac{1}{n^{10}}$$

$\sum \frac{1}{n^{10}}$ converges by p-test. $\sum \frac{1}{n^{10}+4}$ converges by comparison test.

$$\frac{3+\cos^2(n)}{n^2} \le \frac{4}{n^2}$$

So, $\sum \frac{3+\cos^2(n)}{n^2}$ convergent.

\# $\dfrac{n}{n^3-7} = \dfrac{1}{n^2-7/n} \leq \dfrac{1}{n^2-n^2/2} = \dfrac{2}{n^2}$ $\qquad \dfrac{a_n}{1/n^p} \to l.$

since, $7/n \leq n^2/2$ for $n \geq 3$

$\Rightarrow n^2 - 7/n \geq n^2 - n^2/2$

So, $\sum \dfrac{n}{n^3-7}$ convergent by comparison test.

\# $a_n = \dfrac{3n^2}{n^3+5} = \cancel{\dfrac{3}{n+5/n^2} \geq \dfrac{3}{n}}$

$b_n = \dfrac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3n^3}{n^3+5} = 3.$$

By comparison test (limit test), $\sum \dfrac{3n^2}{n^3+5}$ diverges.

\# $a_n = \dfrac{5n^2+7n+5}{9n^{10}+2n+3}$

Let, $b_n = \dfrac{1}{n^8}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(5n^2+7n+5)n^8}{9n^{10}+2n+3} = \frac{5}{9}.$$

So, $\sum a_n$ Convergent as $\sum b_n$ is Convergent.

③

\# $\frac{a_n}{x^n} \to l$ then, $\sum a_n$ & $\sum x_0^n$ converge or diverge together. ($|x|<1$ convergent)

$$a_n = \frac{4^n + n^7 + 4}{17^{2n} + 5^n + 6}. \qquad b_n = \left(\frac{4}{17^2}\right)^n.$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{4^n + n^7 + 4}{17^{2n} + 5^n + 6} \frac{17^{2n}}{4^n}.$$

$$= \lim_{n\to\infty} \frac{1 + \frac{n^7}{4^n} + \frac{4}{4^n}}{1 + \frac{5^n}{17^{2n}} + \frac{6}{17^{2n}}} = 1.$$

and $\frac{4}{17^2} < 1$ so, convergent.

\# $\sum \frac{(n+1)(7^2+1)^n}{7^{2n}}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(n+1)(7^2+1)^n}{7^{2n}}$$

$$= \lim_{n\to\infty} (n+1)\left(1 + \frac{1}{7^2}\right)^n = \infty.$$

So, divergent.

\# $\sum \frac{1}{n(\sqrt{5n-1})}$ $\qquad b_n = \sum \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{n^{3/2}}{n\sqrt{5n-1}} = \lim_{n\to\infty} \sqrt{\frac{n}{5n-1}} = \frac{1}{\sqrt{5}}$$

$\sum b_n$ convergent. so convergent.

$$\sum \frac{n+4^n}{n^2 4^n} = \sum \frac{n}{n^2 4^n} + \sum \frac{1}{n^2} \quad \text{convergent.}$$

$$\le \sum \frac{1}{4^n} \quad \text{convergent.}$$

$$\sum \tan\left(\frac{1}{6n}\right) \qquad \lim_{n\to\infty} \frac{\tan\left(\frac{1}{6n}\right)}{1/6n} = 1.$$

So, $\sum \tan\left(\frac{1}{6n}\right)$ diverges.

$$\# \quad \sum \frac{4n^4+n^2-4n}{9n^{12}-7n^{10}+4} \sim \sum \frac{1}{n^{12-4}} = \sum \frac{1}{n^8}$$

$$\# \quad \sum \frac{5n^5+8n^8}{748n^{13}+9n^5+7} \sim \sum \frac{1}{n^{13-8}} = \sum \frac{1}{n^5}.$$

$$\# \quad \sum \frac{3n}{3n-1} \qquad \lim_{n\to\infty} \frac{3n}{3n-1} = 1 \neq 0.$$

$$\# \quad \sum \frac{1}{n!} \qquad n! = n(n-1)(n-2)\dots 1 \ge n(n-1) \quad \text{for } n \ge 2$$

$$\Rightarrow \frac{1}{n!} \le \frac{1}{n(n-1)} \le \frac{1}{n^2}$$

So, $\sum \frac{1}{n!}$ convergent.