①

Ex Find the slope of the tangent to the curve $y = x^{\sin x}$ at $x = \pi$.

$$\ln(y) = (\sin x) \cdot \ln(x).$$

$$\Rightarrow \frac{1}{y} y' = \cos x \ln(x) + \frac{1}{x} \sin x.$$

$$\therefore y' = y \left(\cos x \cdot \ln(x) + \frac{\sin x}{x} \right)$$

$$y' \Big|_{x=\pi} = \pi^0 \left((-1) \cdot \ln(\pi) \right)$$

$$= \boxed{-\ln(\pi)}$$

$\Rightarrow$ General Exponential fn :-

$$a = \exp(\ln(a)) \qquad [= f^{-1}(f(a))] \qquad \text{for } a > 0$$

$$\Rightarrow a^r = \left[e^{\ln(a)} \right]^r = e^{r \ln(a)} \qquad \text{for } r \text{ rational.}$$

In general, $a^x := e^{x \ln(a)}$, exponential fn with base a.

Prop :- i) $a^{x+y} = a^x \cdot a^y$. ii) $a^{x-y} = a^x / a^y$.

iii) $(a^x)^y = a^{xy}$, iv) $(ab)^x = a^x b^x$.

$$\# \quad \frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{x \ln a}\right) = e^{x \ln(a)} \frac{d}{dx}(x \ln(a))$$

$$= \ln(a) \, e^{x \ln(a)} = a^x \cdot \ln(a)$$

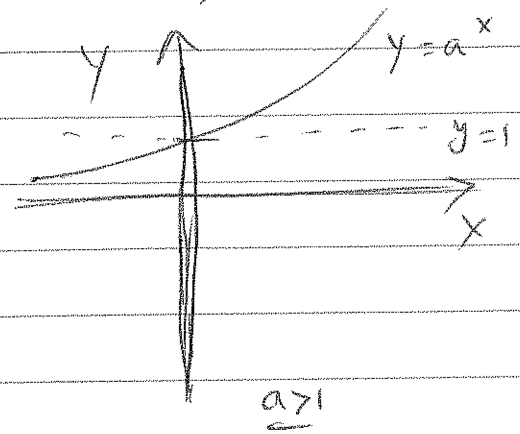
$$\Rightarrow \int a^x \, dx = \frac{1}{\ln(a)} a^x + C. \qquad a \neq 1$$

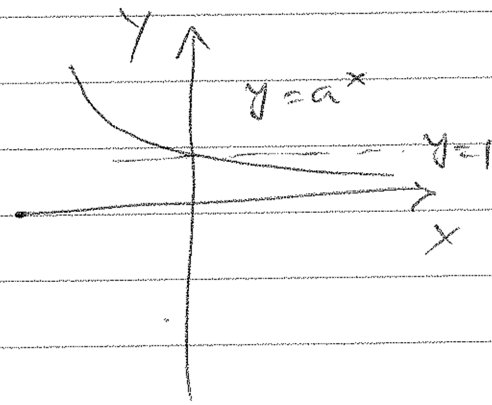
$\int^2 10^t$ — (45) J.S.

Graph:-

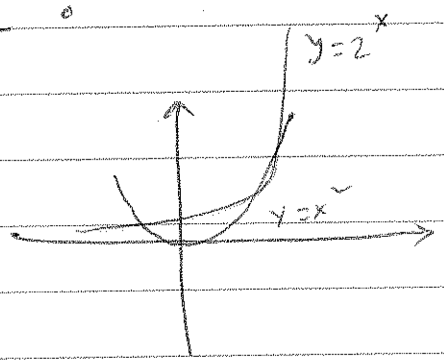
If $a>1$, then $\frac{d}{dx}(a^x) = a^x \ln(a) > 0$; So increasing

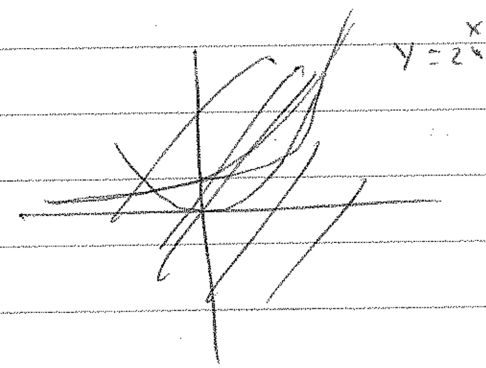
and $0<a<1$, $\frac{d}{dx}(a^x) < 0$, So decreasing.





2^x vs x^2:





2^x grows (and slows) faster than x^2!

$f(x) = 10^x$, $g(x) = x^{10}$

$f(10) = 10^{10}$ $\qquad f(100) = 10^{100}$

$g(10) = 10^{10}$ $\qquad g(100) = (100)^{10} = 10^{20} < f(100)$

(More in L'Hospital Rule)

③

Power Rule:

$$\frac{d}{dx}(x^n) = n x^{n-1}$$

$$\left[\frac{d}{dx}(n^x) = n^x \ln(n) \right]$$

Let, $y = x^n$

$\Rightarrow \ln|y| = n \ln|x|$, $x \neq 0$.

$\Rightarrow \frac{1}{y} y' = \frac{n}{x}$. $\Rightarrow y' = n \frac{y}{x} = n x^{n-1}$.

i) $\frac{d}{dx}\left((f(x))^a\right) = a [f(x)]^{a-1} f'(x)$.

ii) $\frac{d}{dx}\left(a^{f(x)}\right) = a^{f(x)} \ln(a) \cdot f'(x)$.

Find y'; $y = x^{x^2}$

$\ln y = x^2 \ln x$

$\Rightarrow \frac{1}{y} y' = 2x \ln(x) + x^2 \frac{1}{x}$. $= x + 2x \ln(x)$

$\therefore y' = x^{x^2}\left(x + 2x \ln(x)\right)$

[More

$$\int a^{u(x)} \cdot u'(x)\, dx = \frac{a^{u(x)}}{\ln(a)} + C$$

]

④

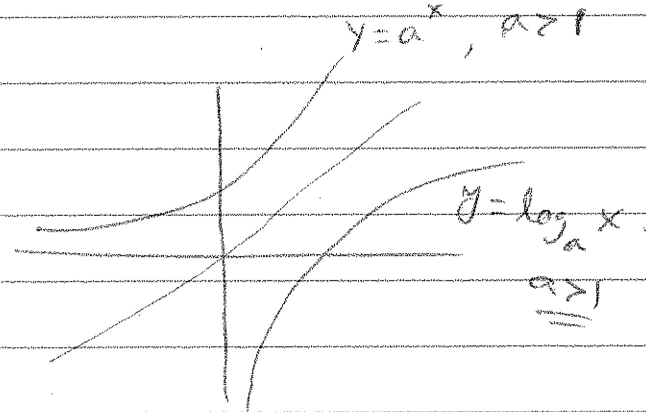
If $a > 0$ and $a \neq 1$, then, $f(x) = a^x$ is 1-1.

$$\left[\ \frac{d}{dx}(a^x) = a^x \ln(a) \ . > \text{or} < 0\ \right]$$

So, inverse fn exists and is called the logarithmic fn with base a. and denoted by $\log_a$

$$\log_a x = y \iff a^y = x.$$

$$\Rightarrow \log_e x = \ln x.$$



$$\begin{cases} \log_a(a^x) = x \\ \& \quad a^{\log_a(x)} = x. \end{cases}$$

$$\frac{d}{dx}\left(a^{\log_a(x)}\right) = \frac{dx}{dx} = 1$$

$$\Rightarrow a^{\log_a(x)} \ln(a) . \frac{d}{dx}\left(\log_a(x)\right) = 1$$

$$\Rightarrow \boxed{\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \ln a}}$$

$3^{2} = 9$

$\Rightarrow \log_3 9 = \log_3 3^2$

$= 2.$

3 ~~what~~ raised to 2 is 9.

$\log_b 49 = \cancel{?}\ 2$

$\Rightarrow \underline{b^2 = 49}$

Connection: $y = \log_a x \Rightarrow a^y = x$. , $x > 0$, $a > 0$

Take ln on both sides,

$$y \ln a = \ln x$$

change of base

$$\Rightarrow \boxed{y = \frac{\ln x}{\quad}}$$

$\log_a a = 1$

Ex. (5) $\frac{d}{dx}\left(\log_{10}(2+\sin x)\right) = \frac{d}{dx}\left[\frac{\ln(2+\sin x)}{\ln 10}\right]$

$$= \frac{1}{\ln 10} \cdot \frac{1}{(2+\sin x)} \frac{d}{dx}(2+\sin x) \qquad \frac{d}{dx}\ln(u)$$

$$= \frac{1}{\ln 10} \, \frac{1}{2+\sin x} \cdot \cos x.$$

Important Observation:-

$$y = \ln x = f(x).$$

$$\Rightarrow f'(x) = 1/x. \quad \text{So, } f'(1) = 1.$$

By definition;

$$f'(1) = \lim_{x\to 0} \frac{f(1+x) - f(1)}{x}.$$

$$= \lim_{x\to 0} \frac{\ln(1+x) - \ln(1)}{x}$$

$$= \lim_{x\to 0} \ln(1+x)^{1/x}.$$

So, $\underline{\lim_{x\to 0} \ln(1+x)^{1/x} = 1}$

$$e = e^{1} = e^{\lim_{x\to 0} \ln(1+x)^{1/x}} = \lim_{x\to 0} e^{\ln(1+x)^{1/x}} \quad \text{(By continuity)}$$

$$= \lim_{x\to 0} (1+x)^{1/x}.$$

So, $\underline{\lim_{x\to 0} (1+x)^{1/x} = e} \cdot \quad \cdots (*)$

Take, $x = 1/n$. then, $(*) \Rightarrow \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$

7.5

22. $\lim_{x \to -\infty} (1.001)^x = \lim_{x \to -\infty} e^{x \ln(1.001)}$

$= 0$ $\qquad \ln(1.001) > 0$

50. $\int \frac{2^x}{1+2^x}\, dx$ $\qquad$ Take $u = 2^x$

$du = 2^x (\ln 2)\, dx$.

$= \int \frac{1}{\ln 2} \frac{du}{1+u} = \frac{1}{(\ln 2)} \ln(|1+u|)$

$= \frac{1}{\ln(2)} \ln(1+2^x)$

$= \underline{\log_2 (1+2^x)}$.

54 $\quad x^y = y^x$.

Take $\ln$ on both sides

$y \ln x = x \ln y$.

$\therefore\ y' \ln x + y/x = \ln y + \frac{x}{y} y'$

$\Rightarrow\ y' = \underline{\dfrac{\ln y - y/x}{\ln x - x/y}}$.

55. $y = \log_{10}\left(1+\frac{1}{x}\right) \Rightarrow 10^y = \left(1+\frac{1}{x}\right)$ $\qquad \underline{x \neq 0}$

$\Rightarrow \frac{1}{x} = 10^y - 1$

$\therefore\ x = \frac{1}{10^y - 1}$

$\therefore\ f^{-1}(x) = \frac{1}{10^x - 1}$ $\qquad x \neq 0$