6.7 Hyperbolic fn :-

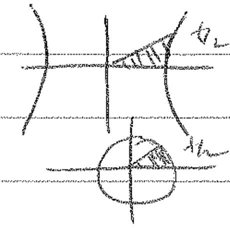
$\sin\theta, \cos\theta \leftrightarrow$ circle, $\theta \in [0, 2\pi)$ $\quad$ $\sinh\theta, \cosh\theta \leftrightarrow$ hyperbola.

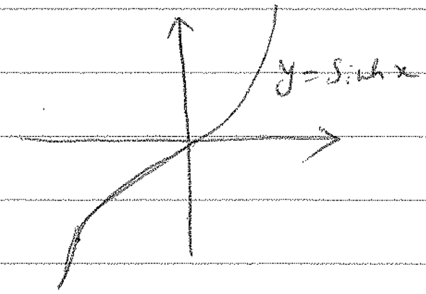
$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \operatorname{cosech} x = \frac{1}{\sinh x}$$

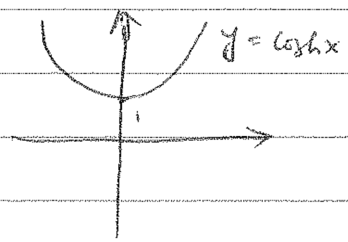
$(\cosh\theta, \sinh\theta)$, $\theta \in \mathbb{R}$

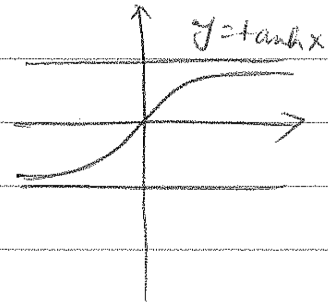
$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$









Prop:- $\sinh(-x) = -\sinh x$, $\quad \cosh(-x) = \cosh x$.

$\cosh^2 x - \sinh^2 x = 1$, $\quad 1 - \tanh^2 x = \operatorname{sech}^2 x$.

$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.

$\Rightarrow \frac{d}{dx}(\sinh x) = \cosh x$, $\quad \frac{d}{dx}(\cosh x) = \sinh x$ $\quad \frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$.

$\frac{d}{dx}(\operatorname{cosech} x) = -\cancel{\coth x}\ \operatorname{cosech} x \coth x$, $\quad \frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$.

Inverse hyperbolic fn :- $y = \sinh^{-1} x \Leftrightarrow \sinh y = x$.

$y = \cosh^{-1} x \Leftrightarrow \cosh y = x \ \& \ y \geq 0$

$y = \tanh^{-1} x \Leftrightarrow \tanh y = x$.

②

$\sinh^{-1}x = \ln\left(x + \sqrt{x^2+1}\right)$, $x \in \mathbb{R}$

$$y = \sinh^{-1}x \Rightarrow \sinh y = x$$

$$\Rightarrow \frac{e^y - e^{-y}}{2} = x$$

$$\Rightarrow e^{2y} - 2xe^y - 1 = 0$$, quadratic in e^y.

$$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$= x \pm \sqrt{x^2+1}$$

But, $e^y > 0 \Rightarrow e^y = x + \sqrt{x^2+1}$

$$\therefore y = \ln\left(x + \sqrt{x^2+1}\right).$$

$\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ $-1 < x < 1$.

$(-1 < x < 1)$

$y \in \mathbb{R}$.

$$y = \tanh^{-1}x \Rightarrow \tanh y = x.$$

$$\Rightarrow \frac{e^y - e^{-y}}{e^y + e^{-y}} = x.$$

$$\Rightarrow (e^{2y} - 1) = x(e^{2y} + 1)$$

$$\Rightarrow e^{2y}(1-x) = 1 + x.$$

$$\therefore e^{2y} = \frac{1+x}{1-x}$$

$$\therefore 2y = \ln\left(\frac{1+x}{1-x}\right)$$

$$\Rightarrow y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$, $-1 < x < 1$

\# $\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}$

$y = \sinh^{-1}x$.

$\Rightarrow \sinh y = x$.

$\frac{d}{dx}(\sinh y) = \frac{dx}{dx} = 1$

$\Rightarrow \cosh y.\ y' = 1 \Rightarrow y' = \frac{1}{\sqrt{1+\sinh^2 y}}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$.

\# $\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}$, $\frac{d}{dx}(\operatorname{cosech}^{-1}x) = -\frac{1}{|x|\sqrt{x^2+1}}$

$\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}}$

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17 $\tanh(\ln x) = \frac{e^{\ln x} - e^{-\ln x}}{e^{\ln x} + e^{-\ln x}}$

$= \frac{x - 1/x}{x + 1/x} = \frac{x^2-1}{x^2+1}$

37 $f(t) = \operatorname{Sech}^2(e^t)$

$\frac{d}{dt} f(t) = -2\operatorname{Sech}^2(e^t)\tanh(e^t)\, e^t$

39. $\frac{d}{dx} G(x) = \frac{-2\sin x}{(1+\cos x)^2}$

47. $\frac{d}{dx}\tan^{-1}(\tanh x) = \frac{\operatorname{Sech}^2 x}{1+\tan^2 x} = \frac{1}{\sinh^2 x + \cosh^2 x}$

$= \frac{4}{2(e^{2x}+e^{-2x})} = \operatorname{Sech}(2x)$

62 $\int \cancel{\tanh x \, dx} = \int \frac{\sinh x}{\cosh x} dx$

let, $\cosh x = u$

$\sinh x \, dx = du$

$= \int \frac{du}{u}$

$= \ln|\cosh x| + c.$

66. $\int_0^1 \frac{1}{\sqrt{1+16t^2}} dt.$

Let, $4t = \sinh u$

$\Rightarrow 4dt = \cosh u \, du.$

$= \frac{1}{4}\int_0^{\sinh^{-1}(4)} \frac{\cosh u \, du}{\sqrt{1+\sinh^2 u}}$

$\cosh^2 u - \sinh^2 u = 1$

$= \frac{1}{4}\int_0^{\sinh^{-1}(4)} du.$

$= \frac{1}{4} \sinh^{-1}(4)$

* $\int \operatorname{Sech}^2(\underbrace{2x - 1/4}_{u}) dx = \frac{1}{2}\tanh(2x - 1/4) + c.$

* $\int_{-\pi/9}^{\pi/9} \sin(5\sinh(6x)) dx. = 0$ as the integrand is an odd fn.

\# $\int_0^{\ln 2} 4e^x \sinh x \, dx = 4\int_0^{\ln 2} e^x\left(\frac{e^x - e^{-x}}{2}\right)dx$

$= 2\int_0^{\ln 2} (e^{2x} - 1) dx = 2\left[\frac{1}{2}e^{2x} - x\right]_0^{\ln 2} = 3 - \ln 4$