$\int_a^b f(x)\,dx$ with f on finite interval and f does not have infinite discontinuity!

Improper: Infinite interval or infinite discontinuity.

I. Infinite intervals:

Defn: If $\int_a^t f(x)\,dx$ exists for every $t \geq a$, then,

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx, \quad \text{provided}$$

the limit is finite.

$\Rightarrow$ Similarly $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx.$

If the limits are divergent, we call the improper integrals to be divergent.

$\Rightarrow$ If both $\int_{-\infty}^a f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ exist,

then, $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx.$

Exm: $\int_1^\infty \frac{\ln x}{x^2}\,dx.$

$$= \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^2}\, dx$$

Let, $\ln x = u$

$\frac{1}{x} dx = du$

$$= \lim_{t\to\infty} \int_0^{\ln t} \frac{u\, du}{e^u}$$

$$= \lim_{t\to\infty} \left[-u e^{-u} \Big|_0^{\ln t} + \int_0^{\ln t} e^{-u}\, du \right]$$

$$= \lim_{t\to\infty} \left[-\ln t\, e^{-\ln t} - e^{-u} \Big|_0^{\ln t} \right]$$

$$= \lim_{t\to\infty} \left[-\frac{1}{t} \ln t \right] - \lim_{t\to\infty} \left(\frac{1}{t} - 1 \right)$$

$$= - \lim_{t\to\infty} \frac{\frac{1}{t}}{1} - 0 + 1 = 1.$$, so converges!

$\int_1^\infty \frac{1}{x^p}\, dx.$

Soln:- Let, $p = 1$ then $\int_1^\infty \frac{1}{x}\, dx = \lim_{t\to\infty} \int_1^t \frac{1}{x} dx$

$$= \lim_{t\to\infty} \left[\ln|x| \right]_1^t$$

$$= \lim_{t\to\infty} \ln|t| = \infty.$$, so the integral diverges.

Let, $p \neq 1$, $\int_1^\infty \frac{1}{x^p}\, dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\, dx$

$$= \lim_{t\to\infty} \left. \frac{x^{-p+1}}{1-p} \right]_1^t$$

$$= \lim_{t\to\infty} \frac{1}{1-p} \left[\frac{1}{t^{p-1}} - 1 \right]$$

If $p>1$, then, $p-1>0$, i.e. $\frac{1}{t^{p-1}} \to 0$

$$\therefore \int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1} \quad \text{if } p>1$$, so the integral converges!

But for $p<1$,

$$\int_1^\infty \frac{1}{x^p}\,dx = \infty$$

so the integral diverges.

So, $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p>1$, & divergent if $\underline{p \leq 1}$.

II Discontinuous Integrads :-

If $f(x)$ is continuous on $(a, b]$ & discont. at a, then,

$$\int_a^b f(x)\,dx = \lim_{c \to a+} \int_c^b f(x)\,dx.$$

Ily for discontinuity at b,

$$\int_a^b f(x)\,dx = \lim_{c \to b-} \int_a^c f(x)\,dx.$$

& $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for discontinuity at $a<c<b$.

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{c\to 0+} \int_c^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{c\to 0+} 2x^{1/2}\Big|_c^1$$

$$= \lim_{c\to 0+} \left(2-2\sqrt{c}\right) = 2$$, so convergent.

\# $\displaystyle\int_0^2 \frac{1}{|1-x|}\,dx$ discontinuity at $x=1$.

$$\frac{1}{|1-x|} = \begin{cases} \dfrac{1}{1-x}, & \text{if } x \le 1 \\ \dfrac{1}{x-1}, & x > 1. \end{cases}$$

So, $$\int_0^2 \frac{1}{|1-x|}\,dx = \lim_{a\to 1-} \int_0^a \frac{1}{1-x}\,dx + \lim_{b\to 1+} \int_b^2 \frac{1}{x-1}$$

$$= \lim_{a\to 1-} -\ln|1-x|\Big]_0^a + \lim_{b\to 1+} \ln|x-1|\Big]_b^2$$

$$= -\lim_{a\to 1-} \ln|1-a| - \lim_{b\to 1+} \ln|b-1|$$

$$= -(-\infty) - (-\infty) = \infty$$, so diverges!

<u>Comparison test:</u> f and g are continuous on $[a,\infty)$ with $0 \le f(x) \le g(x)$, $x \ge a$. then,

i) $\displaystyle\int_a^\infty f(x)\,dx$ converges if $\displaystyle\int_a^\infty g(x)\,dx$ converges.

ii) $\displaystyle\int_a^\infty g(x)\,dx$ diverges if $\displaystyle\int_a^\infty f(x)\,dx$ diverges.

$\int_1^{\infty} e^{-x^2}\,dx$ if it is finite or not!

$$e^{-x^2} \leq e^{-x} \quad \text{on } [1,\infty) \qquad (x^2 \geq x)$$

So, $\int_1^{\infty} e^{-x}\,dx = -e^{-x}\Big|_1^{\infty} = -e^{-1}$, converges.

So, by comparison test, $\int_1^{\infty} e^{-x^2}\,dx$ converges!

$\int_1^{\infty} \frac{\sin^2 x}{x^2}\,dx$

We have, $0 \leq \sin^2 x \leq 1$

so $0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$ for $x \geq 1$

Now, $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges by p-test $(p=2)$

so, $\int_1^{\infty} \frac{\sin^2 x}{x^2}\,dx$ converges!

$\int_1^{\infty} \frac{1}{\sqrt{x^2-1}}\,dx$

$$\frac{1}{\sqrt{x^2-1}} \geq \frac{1}{\sqrt{x^2}} = \frac{1}{x}$$

and $\int_1^{\infty} \frac{1}{x}\,dx$ diverges!

37 $\int_{-1}^{0} \frac{e^{1/x}}{x^3}dx = \lim_{t\to 0^-} \int_{-1}^{t} \frac{e^{1/x}}{x^3}dx$ Let, $e^{1/x} = u$

$-\frac{1}{x^2}e^{1/x}dx = du$

$= -\lim_{t\to 0^-} \int_{e^{-1}}^{e^{1/t}} \ln(u)\, du$

$= \lim_{t\to 0^-} \left[u\ln u \Big|_{e^{1/t}}^{1/e} - u \Big|_{e^{1/t}}^{e^{-1}} \right]$

$= \lim_{t\to 0^-} \left[-\frac{1}{e} - \frac{e^{1/t}}{t} - e^{-1} + e^{1/t} \right]$

$= -\frac{2}{e} - \lim_{t\to 0^-} \frac{e^{1/t}}{t} + \lim_{t\to 0^-} e^{1/t}$

$= -2/e.$

$\int_{0}^{3} \frac{dx}{\sqrt{9-x^2}} = \lim_{t\to 3^-} \int_{0}^{t} \frac{dx}{\sqrt{9-x^2}}$ Let, $x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

$= \lim_{t\to 3^-} \int_{0}^{\sin^{-1}(t/3)} \frac{\cancel{3}\cos\theta}{\cancel{3}\cos\theta} d\theta$

$= \lim_{t\to 3^-} \left[\theta\right]_{0}^{\sin^{-1}(t/3)} = \pi/2.$

$I = \int_{0}^{\infty} \frac{9}{3x+e^{3x}}dx$

$\frac{9}{3x+e^{3x}} \le \frac{9}{e^{3x}}$ for $x \ge 0$

Now, $\int_{0}^{\infty} \frac{9}{e^{3x}}dx = \lim_{t\to\infty} \int_{0}^{t} \frac{9}{e^{3x}} dx = -\lim_{t\to\infty} \frac{9}{3} \left. e^{-3x} \right]_{0}^{t}$

$= \frac{9}{3}.$

By comparison test. I, converges!

$$\# \int_1^{\infty} \frac{dx}{\sqrt{x^6+2}}$$

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$$\frac{1}{\sqrt{x^6+2}} \leq \frac{1}{\sqrt{x^6}} = \frac{1}{x^3}$$

Now, as $\int_1^{\infty} \frac{1}{x^3}\,dx$ converges, the original integral converges;

$$\# \int_{\pi}^{\infty} \frac{3+\cos(4x)}{x^{1/4}}\,dx$$

$$-1 \leq \cos(4x) \leq 1$$

$$\Rightarrow \quad 2 \leq 3+\cos 4x \leq 4$$

So, $\frac{3+\cos(4x)}{x^{1/4}} \geq \frac{2}{x^{1/4}}$.

Now, $\int_{\pi}^{\infty} \frac{2}{x^{1/4}}\,dx$ diverges as $p<1$, so integral diverges!

<u>Limit Test</u> :-

if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L < \infty$ then, $\int_a^{\infty} f(x)\,dx$ and $\int_a^{\infty} g(x)\,dx$

converge or diverge together.

$$\int_{-5}^{5} x \ln|x|\,dx = \lim_{t\to 0-} \int_{-5}^{t} x\ln(-x)\,dx + \lim_{t\to 0+} \int_{t}^{5} x\ln x\,dx$$

$$= \lim_{t\to 0-} \int_{5}^{-t} u \ln u\,du + \lim_{t\to 0+} \int_t^5 x\ln x\,dx \qquad -x = u, \quad -dx = du$$

$$= \lim_{p\to 0+} \int_5^p u\ln u\,du + \lim_{t\to 0+} \int_t^5 x \ln x\,dx$$

$$= 0.$$