11.3 The integral test

It is all about examining the convergence of an infinite series! (NOT to find the exact sum).

Test:

Let, f is continuous, positive, decreasing function on $[1, \infty)$ and $a_n = f(n)$. Then, $\sum a_n$ is convergent iff the improper integral $\int_1^\infty f(x)\,dx$ is convergent.

Note: $\sum_{n=4}^{\infty} \frac{1}{(n-3)^2} \sim \int_4^\infty \frac{1}{(x-3)^2}\,dx$

$\Rightarrow$ f is decreasing for large x.

ExM: $\sum_{n=1}^{\infty} \frac{1}{n^p}$

If $p<0$, $\lim_{n\to\infty} \frac{1}{n^p} = \infty$. If $p=0$, $\lim_{n\to\infty} \frac{1}{n^p} = 1$

So, n^{th} term does not go to 0, $\sum \frac{1}{n^p}$ diverges for $p \le 0$.

Let, $p>0$: $f(x) = \frac{1}{x^p}$ continuous, decreasing and positive on $[1, \infty)$

Now, $\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p>1$ and diverges if $p \le 1$

So, by integral test, $\sum \frac{1}{n^p}$ converges if $p>1$ and diverges if $p \le 1$.

$2 + \frac{2}{4} + \frac{2}{16} + \frac{2}{64} + \cdots$

$a_n = \frac{2}{4^{n-1}}$ $\qquad \sum_{n=1}^{\infty} \frac{2}{4^{n-1}}$ $\qquad \sum a r^{n-1} = \frac{a}{1-r}$

$$= \frac{2}{1-\frac{1}{4}} = \frac{8}{3}.$$

Exm: $\sum \frac{1}{n^{2/3}}$, $p = 2/3 < 1$, so diverges!

$\sum \frac{1}{n^2} = \frac{\pi^2}{6}$, $\quad \int_1^{\infty} \frac{1}{x^2}\,dx = 1.$

So, $\cancel{\sum a_n}$ $\int_1^{\infty} f(x)\,dx \neq \sum_{n=1}^{\infty} f(n).$

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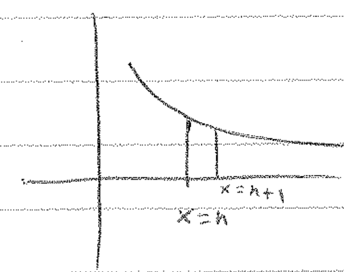
$R_n = s - s_n \qquad f(k) = a_k.$

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_n^{\infty} f(x)\,dx.$$



$\sum_{2}^{\infty} \frac{8}{n \ln n}$

$f(x) = \frac{8}{x \ln x}$ is positive, decreasing and continuous on $[2, \infty)$

$u = \ln x$

Also, $\int_2^{\infty} \frac{8}{x \ln x} = 8 \int_{\ln 2}^{\infty} \frac{du}{u} = 8 \ln u \Big]_{\ln 2}^{\infty} = \infty.$

So, by integral test, $\sum \frac{8}{n \ln n}$ diverges.

Ex14 $\int_1^\infty \frac{3\,dx}{x^2+1} = \left[3\tan^{-1}x\right]_1^\infty$

$= 3\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = \frac{3\pi}{4}$.

So, by integral test, $\sum_{n=1}^{\infty} \frac{3}{n^2+1}$ converges.

23 $\int_1^\infty \frac{e^{1/x}}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{e^{1/x}}{x^2}\,dx$ $e^{1/x} = u$

$-e^{1/x}\frac{1}{x^2}\,dx = du$

$= -\lim_{t\to\infty} \int_e^{e^{1/t}} du$

$= -\lim_{t\to\infty} (e^{1/t} - e) = (e-1)$.

So, $\sum \frac{e^{1/n}}{n^2}$ converges by integral test.

29 $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ $\int_2^\infty \frac{1}{x(\ln x)^p}$ $\ln x = u$

$\frac{1}{x}\,dx = du$

$= \int_{\ln 2}^\infty \frac{1}{u^p}$, convergent if $p > 1$

31 $\sum_{n=1}^{\infty} n(1+n^2)^p$ $\int_1^\infty x(1+x^2)^p$ $1+x^2 = u$

$2x\,dx = du$

$= \frac{1}{2}\int_2^\infty \frac{1}{u^{-p}}\,du$ Convergent, if $-p > 1$

$\Rightarrow p < -1$

32 $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$

$$\int_1^{\infty} \frac{\ln x}{x^p}\,dx$$

$\ln x = u$

$\frac{1}{x}dx = du$

$x = e^u$

(Treat as improper integral)

$$= \int_0^{\infty} \frac{u}{e^{u(p-1)}}\,du$$

$$= \int_0^{\infty} u\,e^{-u(p-1)}\,du$$

$$= -\left[\frac{u\,e^{-u(p-1)}}{p-1}\right]_0^{\infty} + \int_0^{\infty} \frac{e^{-u(p-1)}}{p-1}\,du$$

$$= -\frac{1}{p-1}\left[u\,e^{-u(p-1)}\right]_0^{\infty} - \frac{1}{(p-1)^2}\left[e^{-u(p-1)}\right]_0^{\infty}$$

$$= \begin{cases} \pm\infty & \text{if } p \le 1 \\ \frac{1}{(p-1)^2} & \text{if } p > 1 \end{cases}$$

So, $\displaystyle\sum_{n=1}^{\infty} \frac{\ln}{n^p}$ converges if $p > 1$.