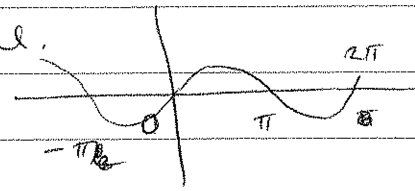
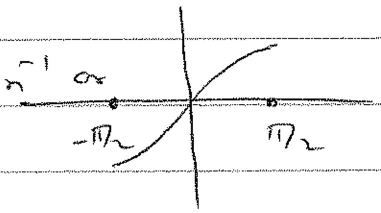
6.6 Inverse Trigonometric functions

$y = \sin x$ is not 1-1, in general.



But can be made 1-1, by restricting the domain

Inverse fn. of Sine is denoted by $\sin^{-1}$ or arcsin.



Defn.: $\sin^{-1}x = y \Leftrightarrow \sin y = x$ & $-\pi/2 \le y \le \pi/2$

\# $\sin^{-1}(\sqrt{3}/2)$ & $\tan(\arcsin 1/3)$.

As, $\sin(\pi/3) = \sqrt{3}/2$, so $\sin^{-1}(\sqrt{3}/2) = \pi/3$. as $\pi/3 \in (-\pi/2, \pi/2)$

Let, $\theta = \sin^{-1}(1/3) \Rightarrow \sin\theta = 1/3. \Rightarrow \cos\theta = +\sqrt{1-\sin^2\theta} = \sqrt{8}/3.$

$\therefore \tan\theta = \frac{1}{2\sqrt{2}}.$

\# $y = \sin^{-1}x$. Then, $\sin y = x$, $-\pi/2 \le y \le \pi/2$

$$\Rightarrow \cos y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$$

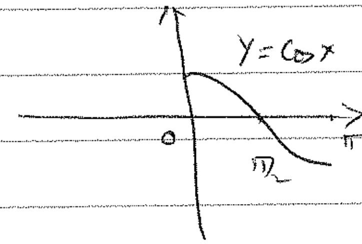
In $-\pi/2 \le y \le \pi/2$, $\cos y \ge 0$. So, $\cos y = +\sqrt{1-\sin^2 y} = \sqrt{1-x^2}$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}, \quad \underline{-1 < x < 1}$$

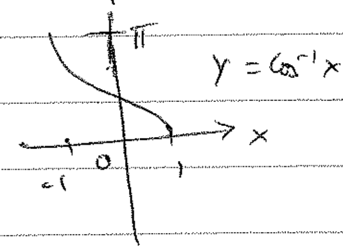
$\cos^{-1}x$: $\quad f(x) = \cos x \quad$ is 1-1 in $\quad 0 \le x \le \pi$

Defn:

$\cos^{-1}x = y \iff \cos y = x \quad$ and $\quad 0 \le y \le \pi$.



$\frac{dy}{dx}$



$y = \cos^{-1}x$.

$\Rightarrow \cos y = x \quad , \quad 0 \le y \le \pi$.

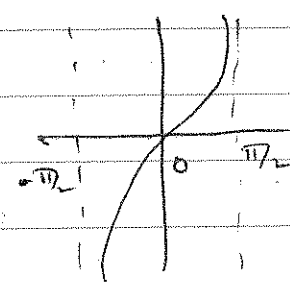
$\therefore -\sin y \frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{1}{\sin y}$

$= -\frac{1}{\sqrt{1-x^2}} \quad , \quad -1 < x < 1$

$f(x) = \tan x \quad$ 1-1 in $\quad -\pi/2 < x < \pi/2$

Defn

$\tan^{-1}x = y \iff \tan y = x \quad$ and $\quad -\pi/2 < y < \pi/2$.



$\frac{dy}{dx}$

$\sec^2 y \frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+x^2}$

$f(x) = \cot x$

$y = \cot^{-1}x \iff \cot y = x \quad$ and $\quad 0 < y < \pi$.

$\frac{d}{dx}\left(\cot^{-1}x\right) = -\frac{1}{1+x^2}$.

$$\frac{d}{dx}(\operatorname{Cosec}^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}, \quad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

So, $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + c$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + c.$$

Exm: $\tan^{-1}x + \cot^{-1}x = \pi/2$.

$f(x) = \tan^{-1}x + \cot^{-1}x$.

$\therefore f'(x) = \dfrac{1}{1+x^2} - \dfrac{1}{1+x^2} = 0$

$\therefore f(x) = \text{Const}$

Now, $f(1) = \tan^{-1}(1) + \cot^{-1}(1) = \pi/2$

So, $\tan^{-1}x + \cot^{-1}x = \pi/2$

26 $g(x) = \sqrt{x^2-1}\,\sec^{-1}x$.

$$g'(x) = \sqrt{x^2-1}\,\frac{1}{x\sqrt{x^2-1}} + \sec^{-1}x\,\frac{x}{\sqrt{x^2-1}}$$

$$= \frac{1}{x} + \frac{x\sec^{-1}x}{\sqrt{x^2-1}}$$

28. $F(\theta) = \sin^{-1}(\sqrt{\sin\theta})$

$$F'(\theta) = \frac{1}{\sqrt{1-\sin\theta}} \cdot \frac{\cos\theta}{2\sqrt{\sin\theta}}$$

$$\frac{dF}{d\theta} = \frac{dF}{dv}\cdot\frac{dv}{du}\cdot\frac{du}{d\theta}$$

$u = \sin\theta$.

$v = \sqrt{\sin\theta} = \sqrt{u}$

35 $$y = \cos^{-1}\left(\frac{b+a\cos x}{a+b\cos x}\right)$$

$$u = \frac{b+a\cos x}{a+b\cos x}.$$

$$= \cos^{-1}(u)$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}.$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u\frac{dv}{dx} - v\frac{du}{dx}}{v^2}.$$

$$= -\frac{1}{\sqrt{1-u^2}} \cdot \frac{(b^2-a^2)\sin x}{(a+b\cos x)^2}$$

$$= \frac{(a^2-b^2)\sin x \cancel{(a+b\cos x)}}{\sqrt{(a^2-b^2)\sin^2 x}\quad (a+b\cos x)^{\cancel{2}}}$$

$$= \frac{\sqrt{a^2-b^2}}{a+b\cos x}.$$

36 $f(x) = \sin^{-1}(e^x)$

$\therefore\ -1 \le e^x \le 1$

so that, $0 \le e^x \le 1$

~~ln~~

$\lim_{x\to 0^+} \ln(x) < x \le \ln(1) \Rightarrow -\infty < x \le 0$

Domain$_f$: $(-\infty, 0]$.

$f'(x) = \frac{e^x}{\sqrt{1-e^{2x}}}$, $e^{2x} \ne 1$

$\Rightarrow x \ne 0$.

Domain$_{f'}$ $= \mathbb{R} - \{0\}$, real line without $\{x=0\}$.

37 $g(x) = \cos^{-1}(3-2x)$

$$-1 \le 3-2x \le 1 \quad \Rightarrow \quad -1 \le 2x-3 \le 1$$
$$\Rightarrow \quad 2 \le 2x \le 4$$
$$\Rightarrow \quad 1 \le x \le 2$$

So, $D_g \equiv [1,2]$

$$g'(x) = -\frac{1}{\sqrt{1-(3-2x)^2}} \cdot (-2) = \frac{2}{\sqrt{(x-1)(8-4x)}}$$

g' is defined for $(x-1)(8-4x) > 0$.

$$\Rightarrow \quad 1 < x < 2$$

$\therefore D_{g'} \equiv (1,2)$

63 $$\int \frac{1+x}{1+x^2}\,dx = \int \frac{1}{1+x^2}\,dx + \int \frac{x\,dx}{1+x^2}$$

Let, $(1+x^2) = u$

$2x\,dx = du$

$$= \tan^{-1}(x) + \frac{1}{2}\int \frac{du}{u}$$

$$= \tan^{-1}x + \frac{1}{2}\ln|1+x^2| + C.$$

68. $$\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx$$

$e^{2x} = u$

$2e^{2x}\,dx = du$

$$= \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}(e^{2x}) + C.$$

69. $$\int \frac{dx}{\sqrt{x}(1+x)} = 2\int \frac{du}{1+u^2}$$

let, $\sqrt{x} = u$

$\frac{1}{2\sqrt{x}}\,dx = du$

$$= 2\tan^{-1}(u) + C$$

$$= 2\tan^{-1}(\sqrt{x}) + C$$