1 Lecture - 1 — Introduction, grading, class test, Attendance

Books :-

1st Part :

1) Thomas & Finney, Calculus and Analytical Geometry.
2) Jain & Iyenger, Advanced Engineering Mathematics. (Narosa)
~~3) Kreyszig, E., Advanced Engineering Mathematics (Wiley)~~.
3) Bartle & Sherbert, Introduction to Real Analysis (Wiley)

2nd Part :

1) Jain & Iyenger
2) Kreyszig, E., Advanced Engineering Mathematics (Wiley)
3) Shepley Ross: Differential Equations. (Wiley)

Syllabus :-

- Single variable derivative
- Multivariate derivative / Calculus
- Vector calculus
- Ordinary diff. equation.

Notation :-

$\mathbb{N}$: natural numbers $\{1, 2, 3, 4, \ldots\}$ mainly for counting

$\mathbb{Z}$: Integers $\{0, \pm 1, \pm 2, \ldots\}$

$\mathbb{R}$: Real numbers

$-\infty$ 0 1 ∞

$\frac{1}{3}, \sqrt{2}$

" "

0.3333... 1.4142....

$\mathbb{Q}$: Rational numbers p/q, no common factor for p & q

$\mathbb{C}$: Complex numbers $a+ib$, $i = \sqrt{-1}$

$\exists$: There exists

$\forall$: For all

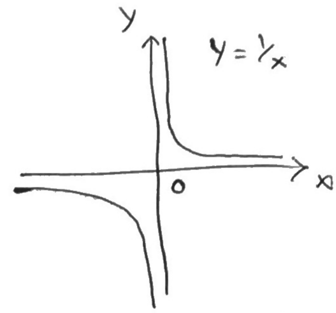
$\in$: belongs to, $\subset$: subset.

Limit :

Calculus (Newton & Leibnitz)
↓
defined function formally
&
Limit

* What is $1/0 \equiv$ undefined.

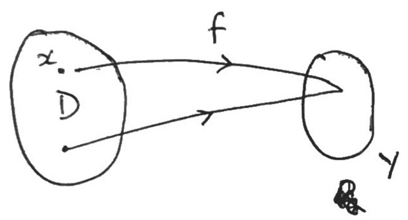
But in limiting sense, $\lim_{x \to 0+} \frac{1}{x} = \infty$



Function :-

$y = f(x)$. variable x.

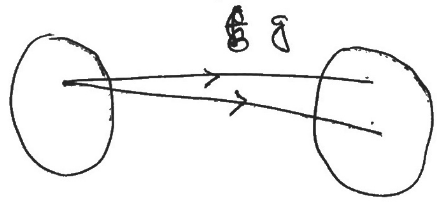
$f : D \to Y$
↑ domain
↑ Co-domain



Range $= \{ y \in Y : \exists\, x \in D$ s.t. $f(x) = y \}$

Range, $R \subseteq$ Co-domain, Y

function $\equiv$ having an unique image for each element in D.



not a function

Injective map :

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ | Vice versa not true

Exm :- $f(x) = \frac{1}{x}$, $x > 0$

Counter Exm $f(x) = x^2$, $x \in [-1, 1]$

Surjective :

For each $y \in Y$, $\exists\, x \in D$ such that $f(x) = y$.

Exm : $f : [-1, 1] \to [0, 1]$, $f(x) = x^2$.

Functions :

1) Polynomial f^{n} — $x, x^2, \dots$

2) trigonometric f^{n} — $\sin, \cos$

3) power fraction functions — $x^{2/3}, x^{1/3}$

4) Exponential f^{n} — $\exp(x)$, $\log(x)$, $\log_{10}(x)$.

Transcendental f^{n} (not algebraic)

$$y^2 - x = 0 \Rightarrow y = \sqrt{x} \quad \text{(Root of a polynomial)}$$

$$a_n(x)\, y^n + a_{n-1}(x)\, y^{n-1} + \dots + a_0(x) = 0$$

Limit :-

$$f(x) = \frac{x^2-4}{x-2}, \quad x \neq 2$$

How it behaves near $x = 2$. (Undefined ~~near~~ at $x=2$)

$$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}} = 4.$$

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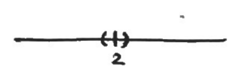
Neighbourhood of a :

$(a-\delta, a+\delta) \equiv$ nbd of a for $\delta > 0$.

$\delta = 0.01$, $a = 2$, $\equiv$ $(1.99, 2.01)$



Limit point / Cluster point :-

$D \subset \mathbb{R}$. (Subset). $a \in \mathbb{R}$ is a limit point of D if for every small $\delta > 0$, $(a-\delta, a+\delta)$ contains at least one point $x \,(\neq a)$ of D.

Exm : $D = (1,2)$

$a_1 = 1.5$ is a limit pt.

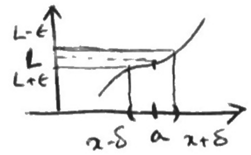
$a_2 = 1, 2$ are limit pt.

$a_3 = 3$ is not a limit pt.

Defn :-

$f: D \to \mathbb{R}$ is a function, $a \in \mathbb{R}$ is a limit point of D.

L is said to be a limit of f at a, if for any (fix) $\epsilon > 0$ $\exists$ a $\delta > 0$ such that,



$$|f(x) - L| < \epsilon \quad \text{for } x \in (a-\delta, a+\delta) - \{a\}$$

$$L - \epsilon < f(x) < L + \epsilon \quad \text{for } 0 < |x-a| < \delta$$

Informally, f(x) gets close to L, for all $x \in D$ sufficiently close to a.

Imp. Result :-

The limit is UNIQUE !

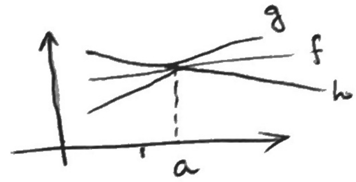
If, $\lim_{x \to a} f(x) = l$ & $\lim_{x \to a} f(x) = m$, then,

$$l = m$$

Sandwich Theorem :-

Let, $g(x) \le f(x) \le h(x)$ $\forall x \in D$, a is a limit pt. of D. and $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = l$, then,

$$\lim_{x \to a} f(x) = l.$$



Exm :-

$$1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2} \quad \forall x \ne 0$$

$$\lim_{x \to 0} \left(1 - \frac{x^2}{4}\right) = \lim_{x \to 0} \left(1 + \frac{x^2}{2}\right) = 1$$

$$\Rightarrow \lim_{x \to 0} u(x) = 1$$

H.W. If $\lim_{x \to a} |f(x)| = 0$, then, $\lim_{x \to a} f(x) = 0$

* Ex $\lim_{x \to 2} (3x+5) = 11.$

Choose $\epsilon > 0$

$$\cancel{3x+5} \quad |f(x) - \ell| = |3x+5-11|$$
$$= |3x-6|$$
$$= 3|x-2| < \epsilon$$
$$\text{if } |x-2| < \epsilon/3 \ (= \delta)$$

Choose δ

See Thomas & Finney for other theoretical Aspects.

One-Sided limits.

Right hand limit

For any $\epsilon > 0$, $\exists\, \delta > 0$ such that,

$$|f(x) - L_1| < \epsilon \quad \text{for } a < x < a+\delta.$$

Notation: $\lim_{x \to a+} f(x) = L_1$.

Left hand limit

For any $\epsilon > 0$, $\exists\, \delta > 0$ s.t.

$$|f(x) - L_2| < \epsilon \quad \text{for} \quad a-\delta < x < a$$

Notation: $\lim_{x \to a-} f(x) = L_2$

If $\lim_{x \to a} f(x) = L$, then, $\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = L$.

H.W Left & right limit $f(x) = \dfrac{x}{|x|}$, at $x = 0$

$$\lim_{x \to 0+} f(x) = 1, \quad \lim_{x \to 0-} f(x) = -1$$

Continuity: $f: D \to \mathbb{R}$, $a \in D$. f is continuous at a if

$$\underline{\lim_{x \to a} f(x) = f(a)}$$

Exm: $f(x) = \begin{cases} x \sin 1/x, & x \neq 0 \\ 0, & \text{else.} \end{cases}$

$$|f(x) - f(0)| = \left| x \sin \frac{1}{x} - 0 \right|$$
$$= |x| \left| \sin \frac{1}{x} \right|$$
$$\leq |x| < \epsilon \quad \text{if } |x| < \epsilon \ (= \delta)$$

$\underline{\lim_{x \to 0} f(x) = f(0)}$, so continuous

Intermediate Value Theorem:- f is continuous in ~~an interval~~ $[a,b]$ and $K \in \mathbb{R}$ such that, $f(a) < K < f(b)$. Then, $\exists\, c \in [a,b]$ such that, $\underline{f(c) = K}$.

Converse is not true

$$f: [0,2] \to \mathbb{R}$$
$$f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 3-x, & 1 < x < 2 \\ 2, & x = 2 \end{cases}$$

$f([0,2]) = [0,2]$, but f is not cont. at 1 & 2.

\# f is cont. on $[a,b]$. Then, f is bounded in $[a,b]$ and $\exists\, c$ & $d \in [a,b]$ such that,

$$f(c) = \sup_{x \in [a,b]} f(x) \quad \text{and} \quad f(d) = \inf_{x \in [a,b]} f(x).$$

(Maxima - Minima property)