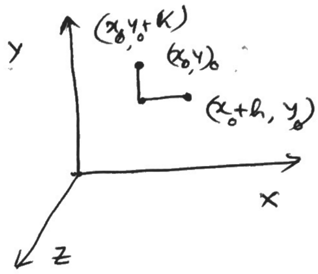
Partial Derivatives :-

$$y = f(x) \quad : \quad \frac{df}{dx} \text{ or } \frac{dy}{dx} \checkmark$$

$$y = f(x,y) \quad : \quad \frac{df}{dx}, \frac{df}{dy} = ? \qquad \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}.$$

Partial derivatives :-



If the limit

$$\lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} \quad \text{exists,}$$

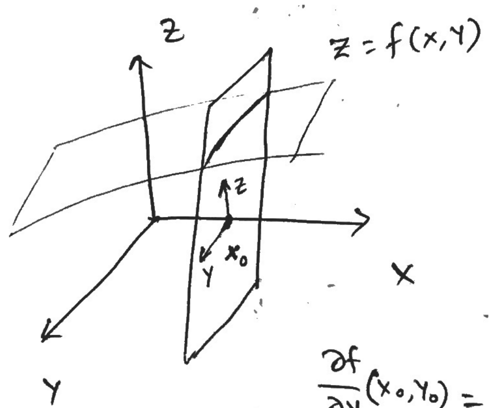
it is called the partial derivative of f w.r.to x at (x_0, y_0) and denoted by

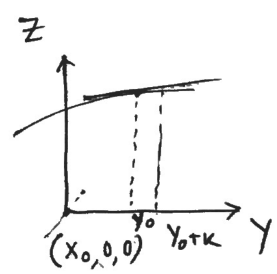
$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0 \\ y=y_0}} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}.$$

Ily,

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{k \to 0} \frac{f(x_0, y_0+k) - f(x_0, y_0)}{k}.$$

Geometric :-





$$\frac{\partial f}{\partial y}(x_0,y_0) = \lim_{k\to 0} \frac{f(x_0, y_0+k) - f(x_0,y_0)}{k}$$

$\frac{\partial f}{\partial y}(x_0, y_0)$ = rate of change of f w.r.to y when x is fixed at x_0

f_y, f_x. Notation

$$\frac{\partial f}{\partial y}(x_0, y_0) = \frac{d}{dy} f(x_0, y) \Big|_{y = y_0}$$

and
$$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{d}{dx} f(x, y_0) \Big|_{x = x_0}.$$

Find f_y and f_x : $f(x,y) = y \sin(xy)$

$$f_x = y^2 \cos(xy)$$

$$f_y = \sin(xy) + xy \cos(xy)$$

1D : differentiability $\Rightarrow$ Continuous

2D : NOT True !

Continuous but P.ds do not exist :

$$f(x,y) = \begin{cases} (x+y)\sin\left(\frac{1}{x+y}\right), & x+y \neq 0 \\ 0, & x+y = 0 \end{cases}$$

Continuous at $(0,0)$.

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \sin\left(\frac{1}{h}\right)$$

do not exist.

Not continuous but P.ds exist :-

$$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^6+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Not continuous : Check on lines $y = mx^3$, $x \neq 0$.

$$f_x(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x} = 0$$

$$f_y(0,0) = 0$$

Sufficient condition for Continuity :-

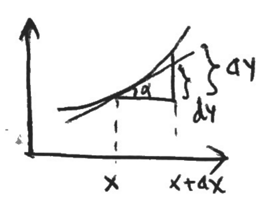
If $f_x(x_0, y_0)$, ~~exists~~ $f_y(x_0, y_0)$ exist, and one of f_x and f_y is bounded in a neighbourhood of (x_0, y_0), then f is continuous.

HW

$$f(x,y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

Not continuous, P.ds exist at $(0,0)$.

Total Derivative (Differentiability)



1D:
$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

or $\quad f(x_0+h) - f(x_0) = h f'(x_0) + \epsilon h$ $\qquad$ $dy = f'(x)\,dx$

where, $\epsilon \to 0$ as $h \to 0$. $\quad$ differential

2D. $\quad z = f(x,y)$,

$$f(x_0+h, y_0+k) = f(x_0,y_0) + h f_x(x_0,y_0) + k f_y(x_0,y_0) + \epsilon_1 h + \epsilon_2 k \quad \cdots (*)$$

With $\epsilon_1, \epsilon_2 \to 0$ as $(h,k) \to (0,0)$

If a function $z = f(x,y)$ satisfies $(*)$, it is said to be differentiable at (x_0, y_0).

~~The~~

Alternative Defn :-

A fn $z = f(x,y)$ is said to be diff. at (x_0, y_0) if the following holds:

$$f(x_0+h, y_0+k) - f(x_0,y_0) = h f_x(x_0,y_0) + k f_y(x_0,y_0) + \epsilon\sqrt{h^2+k^2} \quad \cdots (**)$$

where, $\epsilon \to 0$ as $(h,k) \to (0,0)$.

In limit form:

$$\lim_{(h,k)\to(0,0)} \frac{\left(f(x_0+h, y_0+k) - f(x_0,y_0) - hf_x - kf_y\right)}{\sqrt{h^2+k^2}} = 0$$

Defn :- The expression $(hf_x + kf_y)\big|_{(x_0,y_0)}$ is called the total differential or differential of z at (x_0,y_0), and denoted by dz or df.

$$df = hf_x + kf_y$$

$$\cancel{dz =}$$

$$\cancel{df = hf_x + kf_y}$$

or $$df = f_x \cdot dx + f_y \cdot dy$$

The linear part of the increment.