So,

$$\Delta z = dz + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

with $\epsilon_1, \epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

$(**) \Rightarrow$

$$\lim_{(h,k)\to(0,0)} f(x_0+h, y_0+k) = f(x_0, y_0)$$

Hence, differentiability $\Rightarrow$ continuity.

Theorem:-

- differentiability $\Rightarrow$ P.ds exist.
- P.ds exist $\not\Rightarrow$ differentiable.

ExM:-

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

f is not continuous at $(0,0)$: $y = mx$.

So, f is not differentiable at $(0,0)$

But, $f_x(0,0) = \lim_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = 0$

$f_y(0,0) = 0$.

Or. To be differentiable at $(0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{\left(f(x,y) - f(0,0) - x f_x(0,0) - y f_y(0,0)\right)}{\sqrt{x^2+y^2}} = 0$$

But, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{(x^2+y^2)^{3/2}} = \infty$ along $y = x$

~~Theorem~~

H.W

$$f(x,y) = (xy)^{1/3}$$

Cont., P.ds exists at $(0,0)$. But not differentiable.

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = \infty \text{ along } y = x.$$

Theorem :- If f_x and f_y are continuous at (x_0, y_0), f is differentiable at (x_0, y_0).

Problems :-

i) Find approximate value

ii) Linearize

i) $f(x,y) = \sqrt{xy}$ at $f(2.1, 1.9)$

$$f(x+\Delta x, y+\Delta y) \simeq f(x,y) + \Delta x f_x + \Delta y f_y$$

$$f(2.1, 1.9) = f(2,2) + 0.1 . f_x(2,2) - 0.1 f_y(2,2)$$

$$= 2 + 0.1 \cdot \frac{1}{2} - 0.1 . \frac{1}{2} = 2.$$

Correct Ans: $= \sqrt{3.99} = 1.9975.$

ii) $f(x,y) \simeq f(x_0,y_0) + f_x(x_0,y_0).(x-x_0) + f_y(x_0,y_0).(y-y_0)$

$f(x,y) = x^2 - xy + y^2$ at $(2,2)$.

$$f(x,y) = 4 + 2(x-2) + 2(y-2)$$

Composite function :- (Chain Rule)

$$Z = f(x,y) \quad \text{and} \quad x = g(u,v), \quad y = h(u,v)$$

$$\Rightarrow \quad Z = f(g(u,v), h(u,v)) = F(u,v)$$

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\cancel{\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}}$$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}$$

--- (1)

Exm: $Z = xy, \quad x = u^2 v, \quad y = uv^2$

$$\Rightarrow \quad Z = u^2 v \cdot uv^2 = u^3 v^3$$

Verify (1).

Implicit function :-

$$F(x,y) = 0 \quad (y = g(x))$$

So, $z = F(x,y) = 0 \quad \Rightarrow \quad \frac{dz}{dx} = 0$

$$0 = \frac{dz}{dx} = F_x \cdot \frac{dx}{dx} + F_y \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -F_x / F_y$$

$$\omega = F(x,y,z) = 0 \quad (z = g(x,y))$$

$$\frac{\partial \omega}{\partial x} = F_x \cdot \frac{dx}{dx} + F_y \cdot \frac{dy}{dx} + F_z \cdot \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Higher order derivatives :-

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}(x,y)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{yx}(x,y)$$

In general $f_{xy} \neq f_{yx}$.

Exm $f(x,y) = \begin{cases} \dfrac{xy^3}{x+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

$f_x(0,0) = f_y(0,0) = 0$

$f_x(0,y) = y, \quad f_y(x,0) = 0$

$$f_{xy}(0,0) = \lim_{\Delta x \to 0} \frac{f_y(\Delta x, 0) - f_y(0,0)}{\Delta x} = 0$$

$$f_{yx}(0,0) = \lim_{\Delta y \to 0} \frac{f_x(0,\Delta y) - f_y(0,0)}{\Delta y} = 1$$

or $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{(x^2+y^2)}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Clairaut's theorem :- If f_{xy} ~~(a,b)~~ and f_{yx} are continuous at (x_0, y_0), then,

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$$

- $f_{xxyy} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 f}{\partial y^2}\right)$

- $f_{yyxx} = \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 f}{\partial x^2}\right)$.

Homogeneous function:-

A function $f(x,y)$ is said to be homogeneous of degree n if

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

Exm:-

$$f(x,y) = xy$$

$$f(\lambda x, \lambda y) = \lambda^2 f(x,y)$$

$\Rightarrow$
$$f(x,y) = \frac{xy}{x^4+y^4}$$

$$f(\lambda x, \lambda y) = \frac{1}{\lambda^2}\frac{xy}{x^4+y^4} = \frac{1}{\lambda^2} f(x,y).$$

$\Rightarrow$
$$f(x,y,z) = \frac{xyz}{x^4+y^4}$$

$$f(\lambda x, \lambda y, \lambda z) = \frac{1}{\lambda}\frac{xyz}{x^4+y^4} = \lambda^{-1} f(x,y,z)$$

Euler's theorem:-

If $f(x,y)$ is a homogenous fn of degree n and f has continuous 1st and 2nd order p.ds., then.

i) $$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

ii) $$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f.$$

$$\lambda^n f(x,y) = f(\underbrace{\lambda x}_{x'}, \underbrace{\lambda y}_{y'})$$

$$n\lambda^{n-1} f(x,y) = \frac{\partial f}{\partial x'} \cdot \frac{\partial x'}{\partial \lambda} + \frac{\partial f}{\partial y'} \cdot \frac{\partial y'}{\partial \lambda}$$

$$= x \frac{\partial f}{\partial(\lambda x)} + y \frac{\partial f}{\partial(\lambda y)}$$

$\underline{\lambda = 1}$.

$$\Rightarrow \underline{nf = x f_x + y f_y}$$

$$n(n-1)\lambda^{n-2} f = x \frac{\partial^2 f}{\partial x'^2} \frac{\partial x'}{\partial \lambda} + x \frac{\partial^2 f}{\partial x' \partial y'} \frac{\partial y'}{\partial \lambda}$$

$$+ y \frac{\partial^2 f}{\partial y' \partial x'} \cdot \frac{\partial x'}{\partial \lambda} + y \frac{\partial^2 f}{\partial y'^2} \frac{\partial y'}{\partial \lambda}$$

$\underline{\lambda = 1}$

$$n(n-1) f = x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy}$$

\# Let, $f(x,y) = \dfrac{x^3 + y^3}{(x+y)}$.

P.T. $x f_{xx} + y f_{xy} - f_x = 0$

$$f(\lambda x, \lambda y) = \lambda^2 f(x,y)$$

$$\therefore \quad x f_x + y f_y = 2f$$

$$\Rightarrow \quad f_x + x f_{xx} + y f_{xy} - 2 f_x = 0.$$

$$\Rightarrow \quad \underline{x f_{xx} + y f_{xy} - f_x = 0}$$