Harmonic function :-

A function $z = f(x,y)$ is said to be harmonic if it satisfies Laplace's eqn:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

$$\nabla^2 f = 0 \quad (\Delta f = 0) \quad \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right)\cdot(\quad)$$

Bi-harmonic

$$\Delta\Delta f = 0$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) = 0$$

$$\Rightarrow \quad f_{xxxx} + 2f_{xxyy} + f_{yyyy} = 0$$

Exm: $f(x,y) = e^x \sin y$

$$\frac{\partial f}{\partial x} = e^x \sin y, \quad \frac{\partial^2 f}{\partial x^2} = e^x \sin y$$

$$\frac{\partial f}{\partial y} = e^x \cos y, \quad \frac{\partial^2 f}{\partial y^2} = -e^x \sin y$$

$$\therefore \nabla^2 f = 0 \qquad (\text{nabla}^2)$$

(Potential fn in electromagnetism)

Exm $f(x,y) = x^2 - y^2$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Exm: Constants or linear fn are harmonic.

(x^2-y^2) & xy are harmonic

Taylor's Formula :-

Let, $f(x,y)$ and its P.ds. are continuous upto $(n+1)$th order in some neighbourhood of a point $(x_0, y_0) \in D$. Then, in that nbd,

$$f(x_0+h, y_0+k) = f(x_0,y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0,y_0)$$

$$+ \frac{1}{2!}\left(h^2 f_{xx} + 2hk\, f_{xy} + k^2 f_{yy}\right)\Big|_{(x_0,y_0)}$$

$$+ \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x_0,y_0)$$

$$+ \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f\Big|_{(x_0,y_0)}$$

$$+ \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+ch, y_0+ck)$$

for $c \in (0,1)$.

\# $F(t) = f(x_0+th, y_0+tk)$, $t \in [0,1]$

$$F'(t) = \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)(x_0+th, y_0+tk)$$

$$F''(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0+th, y_0+tk)$$

EXM :- ~~Quadratic~~ Estimate of $f(x,y) = xy$ at $f(0.8, 2.1)$ using quadratic approx. at $f(1,2)$

$$f(x,y) = f(1,2) + (x-1) f_x(1,2) + (y-2) f_y(1,2)$$

$$+ \frac{1}{2!}\left[(x-1)^2 f_{xx} + 2(x-1)(y-2) f_{xy} + (y-2) f_{yy}\right]$$

So, $f(0.8, 2.1) = 2 + 2.(-0.2) + 0.1.1 + \frac{2.(-0.2)(0.1)}{2}$

$= 2 - 0.3 - 0.2$

$= 1.5$

Exm :- 2nd order Taylor approx. of $f(x,y) = 2x^3 + 3y^3 - 4x^2y$ about $(1,2)$. $|R_2| \le B$ $\quad |x-1| < 0.01, \ |y-2| < 0.1$

$$f(x,y) = 2x^3 + 3y^3 - 4x^2y$$

$$f_x = 6x^2 - 8xy \qquad f_{xx} = 12x - 8y$$

$$f_y = 9y^2 - 4x^2 \qquad f_{xy} = -8x$$

$$f_{yy} = 18y$$

$$f_{xxx} = 12, \quad f_{xxy} = -8, \quad f_{xyy} = 0, \quad f_{yyy} = 18.$$

$$f(x,y) = f(1,2) + (x-1) f_x(1,2) + (y-2) f_y(1,2)$$

$$+ \frac{1}{2}\left[(x-1)^2 f_{xx} + 2(x-1)(y-2) f_{xy} + (y-2)^2 f_{yy} \right]$$

$$= 18 - 10(x-1) + 32(y-2) - 2(x-1)^2 - 8(x-1)(y-2)$$

$$+ 18(y-2)^2.$$

$$R_2 = \frac{1}{3!}\left[(x-1)^3 f_{xxx} + 3(x-1)^2(y-2) f_{xxy} + 3(x-1)(y-2)^2 f_{xyy} \right.$$

$$\left. + (y-2)^3 f_{yyy} \right]$$

So, $|R_2| \le \frac{1}{6}\left[(0.01)^3 \cdot 12 + 3(0.01)^2(0.1)(8) + (0.1)^3\, 18 \right].$