~~Curvature~~

Maxima & Minima :-

Let, $f(x,y)$ is defined on a ~~region~~ neighbourhood of (x_0, y_0)

1) f has a local maxima at (x_0, y_0) if $\exists$ a nbd of (x_0, y_0) in which

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$$f(x,y) \le f(x_0, y_0).$$

II) f has a local minima at (x_0, y_0) if $\exists$ a nbd of (x_0, y_0) in which

$$f(x,y) \geq f(x_0, y_0).$$

$\Rightarrow$ The points of maxima / minima are called points of extrema. ~~or stationary point~~.

Theorem (first derivative test)

If $f(x,y)$ has a local maxima/minima at an interior pt (x_0, y_0), and if P.ds exist at (x_0, y_0), then

$$f_x(x_0, y_0) = 0 = f_y(x_0, y_0)$$

Critical point or stationary point :- The points (x,y) for which $f_x(x,y) = 0 = f_y(x,y)$ are called critical pt or stationary pt.

Saddle point :- A differentiable function $z = f(x,y)$ has a saddle pt. at a critical pt (x_0, y_0) if any nbd of (x_0, y_0) there are pts for which both

$$f(x,y) > f(x_0, y_0) \text{ and}$$

$$f(x,y) < f(x_0, y_0)$$

hold. The pt. $(x_0, y_0, f(x_0, y_0))$ is called a saddle pt. of the surface.

Exm: $f(x,y) = x^2 - y^2$

$f_x = 2x = 0$
$f_y = -2y = 0$ $\Rightarrow$ $(x,y) = (0,0)$.

So, ~~the origin~~ the function may have local extrema at $(0,0)$.

But, along $y = 0$, $f(x,0) = x^2 > 0 = f(0,0)$

and along $x = 0$, $f(0,y) = -y^2 < 0 = f(0,0)$

Therefor, the fn. has a ~~local~~ saddle pt. at $(0,0)$.

Theorem :- (2nd derivative test):

Suppose $f(x,y)$ and its 1st and 2nd order P.ds. are continuous at a critical pt. (x_0, y_0). Then,

i) f has a local maxima at (x_0, y_0) if $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (x_0, y_0)

ii) f has a local minima at (x_0, y_0) if $f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (x_0, y_0)

iii) f has a saddle pt. at (x_0, y_0) if $f_{xx} f_{yy} - f_{xy}^2 < 0$.

iv) The test is inconclusive if $f_{xx} f_{yy} - f_{xy}^2 = 0$ at (x_0, y_0).

ExM: $f(x,y) = xy^2$

$f_x = y^2 = 0 \Rightarrow y = 0$
$f_y = 2xy = 0 \Rightarrow x = 0$ $\Rightarrow (0,0)$ is a critical pt.

$f_{xx} = 0, \quad f_{yy} = 2x. \quad f_{xy} = 2y$

$\Rightarrow \quad f_{xx} f_{yy} - f_{xy}^2 = 0.$ Inconclusive.

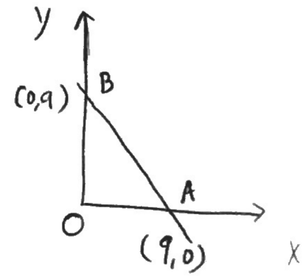
ExM: $f(x,y) = xy$

$f_x = y$
$f_y = x$ $= 0 \Rightarrow (0,0)$ is a critical pt.

$f_{xx} = 0, \quad f_{yy} = 0, \quad f_{xy} = 1$

$f_{xx} f_{yy} - f_{xy}^2 = -1 < 0$ So, f has a saddle pt. at $(0,0)$.

ExM:- Find the absolute maximum or minimum of $f(x,y) = 2 + 2x + 2y - x^2 - y^2$ on the triangle $x = 0, y = 0,$ $x + y = 9$.



$\Rightarrow$ f is differentiable, so f can be extremum at the interior pts or along the boundary.

$f_x = 2 - 2x = 0$
$f_y = 2 - 2y = 0$ $\Rightarrow x = 1, y = 1$

$f_{xx} = -2, \quad f_{yy} = -2, \quad f_{xy} = 0$

So, $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 = 4 > 0$

Therefore, f has a local ~~minima~~ maxima at $(1,1)$

and $f(1,1) = 4$.

We need to check along the boundary.

Along OA: $y = 0 \Rightarrow$

$g(x) = f(x,0) = 2 + 2x - x^2$, $0 \le x \le 9$

~~$f'(x)$~~

$g'(x) = 2 - 2x \Rightarrow x = 1$

$g''(x) = -2 < 0$

Therefore g has a local max at 1 and $g(1) = 3$.

Max along the bdry $= \max\{g(0), g(1), g(9)\} = g(1) = 3$.

Min " " " $= \min\{g(0), g(9), g(1)\} = -61$.

Along OB: $x = 0 \Rightarrow$

$h(y) = f(0,y) = 2 + 2y - y^2$

$\therefore h'(y) = 2 - 2y = 0 \Rightarrow y = 1$

$h''(y) = -2 < 0$

So, h has a local max at 1. & $h(1) = 3$.

Along AB: $y = -x + 9$

So, $K(x) = f(x, -x+9) = 2 + 2x - 2(x-9) - x^2 - (x-9)^2$

$= -2x^2 + 18x - 61$

$\therefore K'(x) = -4x + 18 = 0 \Rightarrow x = 9/2$

$K''(x) = -4 < 0$ $\qquad y = 9/2$

So, K has max at $9/2$ $= K(9/2) = -41/2$

So, the max is $f(1,1) = 4$

and the min is $f(0,9) = f(9,0) = -61$.

Exm: Find the absolute max/min of

$f(x,y) = 3x^2 + y^2 - x$ over $2x^2 + y^2 \le 1$.

$\Rightarrow$

$$\left.\begin{aligned} f_x &= 6x - 1 = 0 \\ f_y &= 2y = 0 \end{aligned}\right\} \Rightarrow x = 1/6,\ y = 0$$

The critical pt is $\left(\frac{1}{6}, 0\right)$.

$f_{xx} = 6$, $f_{yy} = 2$, $f_{xy} = 0$

$\Rightarrow$ $f_{xx} > 0$, $f_{xx} f_{yy} - f_{xy}^2 = 12 > 0$

Hence, f has a local minima at $\left(\frac{1}{6}, 0\right)$, $f\left(\frac{1}{6}, 0\right) =$

Also, $\left(\frac{1}{6}, 0\right)$ is inside the region $2x^2 + y^2 \le 1$

We need to test along the boundary: $\underline{2x^2 + y^2 = 1}$

$\Rightarrow y^2 = 1 - 2x^2$. $\quad -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

$$g(x) = f(x, \pm\sqrt{1-2x^2}) = 3x^2 + (1 - 2x^2) - x$$
$$= x^2 - x + 1, \quad -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$$

$\therefore g'(x) = 2x - 1 = 0 \quad x = 1/2$

$\therefore g''(x) = 2 > 0$.

So, g has min. at $x = 1/2$ and $g\left(\frac{1}{2}\right) = 3/4$.

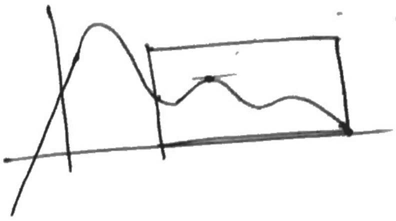
Also, $g\left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{2} + \frac{1}{\sqrt{2}} + 1 = \frac{3}{2} + \frac{1}{\sqrt{2}}$

$g\left(+\frac{1}{\sqrt{2}}\right) = \frac{1}{2} - \frac{1}{\sqrt{2}} + 1 = \frac{3}{2} - \frac{1}{\sqrt{2}}$

So, min. is $f\left(\frac{1}{6}, 0\right) = -\frac{1}{12}$, max is $f\left(-\frac{1}{\sqrt{2}}, 0\right) = \frac{3 + \sqrt{2}}{2}$.

Lagrange Multipliers :-

Constrained Optimization



Max/Min $f(x,y,z)$

with the condition :

$$\begin{cases} g(x,y,z) = 0 \\ h(x,y,z) = 0 \\ \vdots \end{cases}$$

Conside the auxiliary fn:

$$F(x,y,z,\lambda_1,\lambda_2,\lambda_3 \ldots) = f + \lambda_1 g + \lambda_2 h + \cdots$$

Now, by necessary condn. of extrema

$$\frac{\partial F}{\partial x} = 0 = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z}.$$

λ_i are called Lagrange's multipliers.

Exm: Find the max of $x^2+y^2+z^2$ subject to $xyz = a^3$.

Conside the auxiliary function

$$F(x,y,z,\lambda) = x^2+y^2+z^2 + \lambda(xyz - a^3)$$

For extremum, $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$

$$\Rightarrow 2x + \lambda yz = 0, \quad 2y + \lambda xz = 0, \quad 2z + \lambda xy = 0$$

$$\Rightarrow \lambda xyz = -2x^2 = -2y^2 = -2z^2$$

$$\Rightarrow x^2 = y^2 = z^2 \quad \Rightarrow x^6 = a^6 = y^6 = z^6$$

So, (a,a,a), $(a,-a,-a)$, $(-a,a,-a)$, $(-a,-a,a)$ are the solutions.

The value of the function at each of these pts $= 3a^2$

$$A = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

is positive definite & $f_{xx}, f_{yy}, f_{zz} > 0$

So, the fn has local minima at all these pts.

Tut 3

27. Extreme value of $f(x,y) = x^2+2y^2$ on $x^2+y^2=1$.

Consider $F(x,y) = (x^2+2y^2) + \lambda(x^2+y^2-1)$

For extremum, $\frac{\partial F}{\partial x} = 0 = \frac{\partial F}{\partial y}$

$\Rightarrow \quad 2x(1+\lambda) = 0, \quad 2(2+\lambda)y = 0$

So, either $x=0$ or $\lambda=-1$ and $y=0$ or $\lambda=-2$

Let, $x=0 \Rightarrow y=\pm1, \lambda=-2$

or $\lambda=-1 \Rightarrow y=0, x=\pm1$

So, $(\pm1,0)$ and $(0,\pm1)$ are the solutions.

At these pts. the values are $f(\pm1,0) = 1$: min

$f(0,\pm1) = 2$: max

28 Find the pts. on $x^2+y^2+z^2=4$ closest & farthest from $(3,1,-1)$

Max or Min $f(x,y,z) = (x-3)^2 + (y-1)^2 + (z+1)^2$ s.t.

$$x^2+y^2+z^2=4.$$

Consider $F(x,y,z,\lambda) = f + \lambda(x^2+y^2+z^2-4)$

For extremum, $\frac{\partial F}{\partial x} = 0 = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z}$

$\Rightarrow$ $\cancel{2}(x-3) + \cancel{2}x\lambda = 0$, $\cancel{2}(y-1) + \cancel{2}\lambda y = 0$

& $\cancel{2}(z+1) + \cancel{2}z\lambda = 0$

$\therefore$ $x = \frac{3}{1+\lambda}$, $y = \frac{1}{1+\lambda}$, $z = \frac{-1}{1+\lambda}$.

$\underline{\lambda \neq -1}$: $x^2+y^2+z^2=4 \Rightarrow \lambda = -1 \pm \frac{\sqrt{11}}{2}$.

$\therefore$ $\lambda = -1 + \frac{\sqrt{11}}{2} \Rightarrow \left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)$ A

$\lambda = -1 - \frac{\sqrt{11}}{2} \Rightarrow \left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$ B

Check $f(x,y,z)$: A is closest, B is farthest.

29 Let, x = length, y = width, z = height

Max : $V = xyz$ subject to

$2xz + 2yz + xy = 12$

Consider $F(x,y,z,\lambda) = xyz + \lambda(2xz+2yz+xy-12)$

For extremum, $\frac{\partial F}{\partial x} = 0 = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z}$.

$$\Rightarrow \quad yz + \lambda(2z+y) = 0$$
$$xz + \lambda(2z+x) = 0$$
$$xy + \lambda(2x+2y) = 0$$

$\&$ $2xz + 2yz + xy = 12$

If $\lambda = 0$: $xy = yz = zx = 0$, a contradiction.

So, $\lambda \neq 0$. $xyz = -\lambda(2xz + xy) = -\lambda(2zy + xy)$

$$\Rightarrow z(x-y) = 0$$

$z \neq 0$, $(V \neq 0)$ So, $x = y$

Also, $xyz = -\lambda(2yz + xy) = -\lambda(2xz + 2yz)$

$$\Rightarrow x(y - 2z) = 0$$

$x \neq 0 \Rightarrow y = 2z = x$.

$4z^2 + 4z^2 + 4z^2 = 12 \Rightarrow z^2 = 1 \Rightarrow z = 1$.

$\therefore x = 2 = y$

Hence, $V = xyz$
$= 4$.