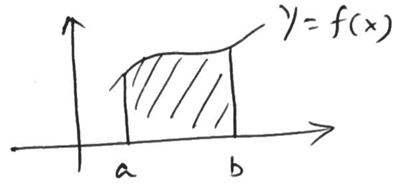
Multiple Integrals :-

$$\int_a^b f(x)\,dx$$



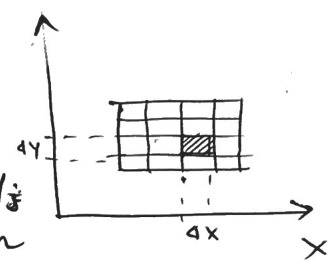
f: piecewise continuous

Domain: bounded, unbounded, infinite intervals.

* Extend it for ~~de~~ $f(x,y)$ in $Q = \{(x,y) : a \le x \le b,\ c \le y \le d\}$

Conside two partition of $[a,b]$ and $[c,d]$

then, $P_1 \times P_2$ would be a partition of Q.



A small piece of area would be $\Delta A_{ij} = \Delta x_i \cdot \Delta y_j$

$i = 1, \dots, n$

$j = 1, \dots, m$

Conside the sum, $S_{nm} = \sum_i^n \sum_j^m f(x_i, y_j)\, \Delta A_{ij}$

If $\lim\limits_{\substack{\Delta A_{ij} \to 0 \\ n,m \to \infty}} S_{nm}$ exists, the limit is called the double integral, and is denoted by

$$\iint_Q f(x,y)\,dx\,dy \quad \text{or} \quad \iint_Q f(x,y)\,dA$$

Note: If $f(x,y)$ is continuous on Q, then the double integral exists.

If $f(x,y)$ is non-negative, we may interpret the double integral of f over a region D as the volume of the solid prism bounded by D and the surface $z = f(x,y)$.

Iterated or Repeated Integrals

$$R = \{ (x,y) \mid a \le x \le b,\ c \le y \le d \}$$

$$\int_a^b \int_c^d f(x,y)\, dx\, dy$$ Double integral

$$I_1 = \int_a^b \left[\int_c^d f(x,y)\, dy \right] dx$$

& $$I_2 = \int_c^d \left[\int_a^b f(x,y)\, dx \right] dy.$$

Iterated integrals

Fubini's theorem (Part 1)

If $f(x,y)$ is continuous on the rectangle R, then

$$\iint_R f(x,y)\, dA = I_1 = I_2.$$

EXM :- $R = [-1,1] \times [0, \pi/2]$

$$\iint_R (x \sin y - y e^x)\, dx\, dy$$

$$I_2 = \int_0^{\pi/2} \left[\int_{-1}^{1} (x \sin y - y e^x)\, dx \right] dy$$

$$= \int_0^{\pi/2} \left[\frac{x^2}{2} \sin y - y e^x \right]_{-1}^{1} dy$$

$$= -\int_0^{\pi/2} y\left(e - \frac{1}{e}\right) dy = -\frac{1}{2}\frac{\pi^2}{4}\left(e - \frac{1}{e}\right)$$

EXM :

1. $$f(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2}$$

$$\int_0^1 \int_0^1 f(x,y)\, dx\, dy = -\pi/4$$

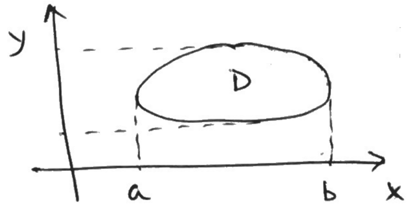
$$\int_0^1 \int_0^1 f(x,y)\, dy\, dx = \pi/4$$

2. $$f(x,y) = \begin{cases} \dfrac{x-y}{(x+y)^3}, & x, \\ 0, & \end{cases}$$

$$\underbrace{\int_0^1 \int_0^1 f\, dx\, dy}_{= -1/2} \neq \underbrace{\int_0^1 \int_0^1 f\, dy}_{= 1/2}$$

Check I_1. H.W. $\displaystyle\int_0^1\int_1^3 \frac{dx\,dy}{(ax+by)^2}$

General Region:



i) $a \le x \le b$, $g_1(x) \le y \le g_2(x)$

ii) $c \le y \le d$, $h_1(y) \le x \le h_2(y)$.

Fubini's theorem (Part 2)

Let $f(x,y)$ is continuous on D.

i) If D is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 & g_2 continuous on $[a,b]$, then

$$\iint_D f(x,y)\,dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx.$$

ii) If D is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$ and h_1, h_2 continuous on $[c,d]$, then.

$$\iint_D f(x,y)\,dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy.$$

$x,y>0$

else

dydx

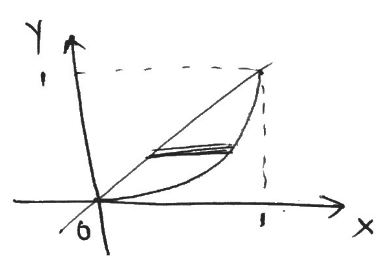
||

$\frac{1}{2}$

EXM:- Change the order of integration ~~and evaluate~~:

$$I = \int_0^1\left[\int_{x^2}^{x} f(x,y)\,dy\right] dx.$$

$$I = \int_0^1\left[\int_y^{\sqrt{y}} f(x,y)\,dx\right] dy.$$

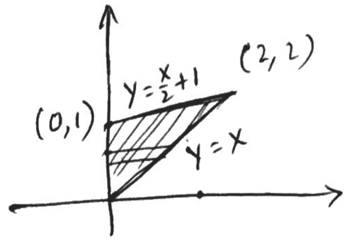


Exm:

$$\iint_D (x+y)^2\,dx\,dy \quad : \text{D is bounded by lines}$$

Joining $(0,0)$, $(0,1)$ & $(2,2)$

$$\int_0^2 \left[\int_x^{x/2+1} (x+y)^2\, dy \right] dx$$



$$= \int_0^2 \left[\frac{(x+y)^3}{3} \right]_0^y dy$$

See class note!

$$= \int_0^2 \left(\frac{8y^3}{3} - \frac{y^3}{3} \right) dy = \frac{7}{3}\left[\frac{y^4}{4}\right]_0^2 = \frac{16\cdot 7}{12} = \frac{4\cdot 7}{3}$$

Change of variables :-

$$\int_0^{1} \frac{dx}{\sqrt{1+x^2}} \qquad x = \tan\theta, \quad dx = \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{\sec\theta} = \int_0^{\pi/4} \sec\theta\, d\theta$$

$$= \Big[\ln |\sec\theta + \tan\theta| \Big]_0^{\pi/4}$$

$$= \ln(1+\sqrt{2})$$

\# let, $x = x(u,v)$ & $y = y(u,v)$ are continuously differentiable fns that maps 1-1 from D_1 in u-v plane onto D in x-y plane.

Then,

$$\iint_D f(x,y)\,dx\,dy = \iint_{D_1} f(x(u,v), y(u,v))\,|J|\,du\,dv$$

where,

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = x_u y_v - x_v y_u$$

is called the Jacobian of the transformation.

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Exm:-

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

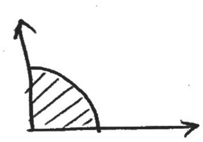
$$J(r,\theta) = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

So, $$\iint_D f(x,y)\,dx\,dy = \iint_{D_1} F(r,\theta)\,r\,dr\,d\theta.$$

T4/17.

$$\iint_D xy\,dA$$



$$= \int_0^a \int_{\theta=0}^{\pi/2} r^2\cos\theta\sin\theta \cdot r\,dr\,d\theta.$$

$$= -\frac{1}{2}\int_0^a r^3 \left[\frac{\cos(2\theta)}{2}\right]_0^{\pi/2} \cancel{d\theta}\,dr$$

$$= -\frac{1}{4}\int_0^a r^3(-1-1)\,dr = \frac{1}{2}\,\frac{a^4}{4} = \frac{a^4}{8}.$$

Triple Integral :-

Analogously to ~~dat~~ double integral, we define triple integral

$$\int_V f(x,y,z)\,dv = \iiint_V f(x,y,z)\,dx\,dy\,dz.$$

$$\int_V f(x,y,z)\,dv = \lim_{\Delta V_{i,j,k}\to 0} \sum_{i,j,k} f(x_i, y_j, z_k)\,\Delta V_{ijk},$$

it exists.

Volume

$$\int_V dv \quad \text{or} \quad \iiint_V dx\,dy\,dz.$$

Exm:

Tut 4

21. $$\int_0^1\int_0^1\int_0^1 xyz\,dz\,dy\,dx$$

$$= \left(\int_0^1 x\,dx\right)\cdot\left(\int_0^1 y\,dy\right)\left(\int_0^1 z\,dz\right)$$

$$= \left[\frac{x^2}{2}\cdot\frac{y^2}{2}\cdot\frac{z^2}{2}\right]_0^1 = \frac{1}{8}.$$

~~23~~ 22 $$\int_0^1\int_0^1\int_{z=\sqrt{x^2+y^2}}^{2} xyz\,dz\,dy\,dx$$

$$\int_{z=\sqrt{x^2+y^2}}^{2} xyz\,dz = \left[\frac{1}{2}xyz^2\right]_{\sqrt{x^2+y^2}}^{2}$$

$$= \frac{1}{2}xy(4-x^2-y^2)$$

$$\int_{y=0}^{1}\left(2xy - \frac{1}{2}x^3y - \frac{1}{2}xy^3\right) dy$$

$$= \cancel{x - \frac{1}{2}x^2}\, x - \frac{1}{4}x^3 - \frac{1}{8}x = \frac{7}{8}x - \frac{1}{4}x^3.$$

& $$\int_{x=0}^{1}\left(\frac{7}{8}x - \frac{1}{4}x^3\right)dx = \left[\frac{7}{10}x^2 - \frac{1}{16}x^4\right]_0^1 = \frac{3}{8}.$$

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$$\int_{x=0}^{1}\int_{y=x}^{1}\int_{z=y}^{1} x\,dz\,dy\,dx.$$

$$\int_{z=y}^{1} x\,dz = x(1-y)$$

$$\int_{y=x}^{1} x(1-y)\,dy = x\left[-\frac{(1-y)^2}{2}\right]_x^1 = +\frac{x(1-x)^2}{2} = +\frac{x(1-2x+x^2)}{2}$$

$$= +\frac{x}{2} - x^2 + \frac{x^3}{2}.$$

So, $$\int_0^1\left(\frac{x}{2} - x^2 + \frac{x^3}{2}\right)dx = \frac{1}{4} - \frac{1}{3} + \frac{1}{8} = \frac{1}{24}.$$

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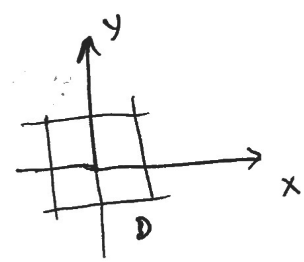
$$V = \int_{-1}^{1}\int_{-1}^{1}\int_{z=0}^{x^2+xy} dz\,dy\,dx$$



$$= \int_{-1}^{1}\int_{-1}^{1}(x^2\cancel{8} + xy)\,dy\,dx$$

$$= \int_{-1}^{1}\left[x^2y + \frac{xy^2}{2}\right]_{-1}^{1} dx$$

$$= \int_{-1}^{1} 2x^2 = \frac{2}{3}\left[x^3\right]_{-1}^{1} = \frac{4}{3}.$$