Scalar & Vector field :-

If we assign a number to every point $P(x,y,z)$ in a domain, we get a scalar function or scalar field.

$$f = f(x,y,z).$$

$f: \mathbb{R}^3 \to \mathbb{R}$

Exm: $D = \mathbb{R}^3$.

$f(x,y,z)$ = distance from origin

$$= \sqrt{x^2+y^2+z^2}$$

If for every point in the domain, we get a vector, the function is called a vector field.

$$g: \mathbb{R}^3 \to \mathbb{R}^3$$

$$g(x,y,z) = (g_1(x,y,z), g_2(x,y,z), g_3(x,y,z))$$

$$\text{or} \quad g_1\hat{i} + g_2\hat{j} + g_3\hat{k}.$$

Exm:

$$g(x,y,z) = (x,y,z)$$
$$= x\hat{i} + y\hat{j} + z\hat{k}.$$

Level surfaces :- Let $f: \mathbb{R}^3 \to \mathbb{R}$ is a scalar field/function.

~~Let~~ Then, $f(x,y,z) = c$ defines the equation of a surface and is called a level surface of the function.

Exm:

$\cancel{x+f}$

$$f(x,y,z) = x^2+y^2+z^2 = 1$$
$$x^2+y^2+z^2 = k^2$$
$$\vdots$$

Gradient of a scalar field :-

The gradient of a differentiable function $f(x,y,z)$ is denoted by ∇f or $\text{grad}(f)$, and defined by

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

or

$$= f_x \hat{i} + f_y \hat{j} + f_z \hat{k}.$$

Recall:

Differentiability:

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - \boxed{h f_x - k f_y}}{\sqrt{h^2+k^2}} = 0$$

$$(h,k)\cdot(f_x, f_y) = (h,k)\cdot \nabla f$$

$$(h\hat{i}+k\hat{j})\cdot(f_x\hat{i}+f_y\hat{j})$$

Exm:

1) $f(x,y) = x+xy$, at $(1,1)$

$$f_x = 1+y, \quad f_y = x$$

$$\nabla f = (1+y)\hat{i} + x\hat{j}.$$

$$\nabla f\big|_{(1,1)} = 2\hat{i} + \hat{j}.$$

11) $f(x,y,z) = xyz^2$ at $(1,0,1)$

$f_x = yz^2, \quad f_y = xz^2, \quad f_z = 2xyz$

$\nabla f = yz^2\,\hat{i} + xz^2\,\hat{j} + 2xyz\,\hat{k}$ #

$\nabla f\big|_{(1,0,1)} = \hat{j} \equiv (0,1,0)$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ & $|\vec{r}| = r$

Prove, $\text{grad}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$.

$r^2 = x^2+y^2+z^2$

$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = y/r \quad \frac{\partial r}{\partial z} = z/r.$

$\text{grad}\left(\frac{1}{r}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$

$= \hat{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + \hat{j}\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right) + \hat{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$

$= -\hat{i}\,\frac{x}{r^3} - \hat{j}\,\frac{y}{r^3} - \hat{k}\,\frac{z}{r^3}$

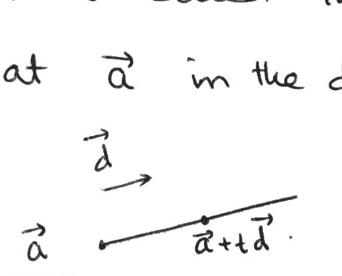
$= -\frac{\vec{r}}{r^3}$

Directional derivative:

Let, $f(x,y,z)$ is a scalar field. Let, $\vec{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$ be any unit vector and $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ be a point. Exm:

If $\displaystyle\lim_{t\to 0} \frac{f(a_1+td_1,\, a_2+td_2,\, a_3+td_3) - f(a_1,a_2,a_3)}{t}$

exists, it is called the directional derivative of f at $\vec{a}$ in the direction of $\vec{d}$.



$$D_{\vec{d}} f(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a} + t\vec{d}) - f(\vec{a})}{t}$$

\# $$D_{\vec{d}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{d} \qquad (\vec{d} \text{ is a unit vector})$$

Let, $$g(t) = f(a_1 + td_1, a_2 + td_2, a_3 + td_3)$$

$$\therefore \quad g'(0) = \cancel{\lim_{t \to 0}} \lim_{h \to 0} \frac{g(h) - g(0)}{h}$$

$$= D_{\vec{d}} f(\vec{a})$$

$$g(t) = f(x, y, z), \quad \begin{aligned} x &= a_1 + td_1 \\ y &= a_2 + td_2 \\ z &= a_3 + td_3 \end{aligned}$$

$$g'(t) = \frac{dg}{dt} = f_x \cdot x_t + f_y \cdot y_t + f_z \cdot z_t$$

$$= (f_x, f_y, f_z) \cdot (d_1, d_2, d_3)$$

$$= \nabla f(x, y, z) \cdot \vec{d}$$

So, $$g'(0) = \nabla f(\vec{a}) \cdot \vec{d}$$

Exm: Directional derivative of $f(x,y,z) = xyz$ at $(1,1,1)$ in the direction of $(3\hat{i} + 4\hat{j} - 5\hat{k})$

$$\nabla f = yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}$$

$$\vec{d} = \frac{3\hat{i} + 4\hat{j} - 5\hat{k}}{\sqrt{3^2 + 4^2 + 5^2}} = \frac{3\hat{i} + 4\hat{j} - 5\hat{k}}{5\sqrt{2}}$$

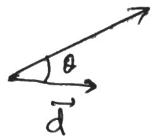
$$\therefore \quad D_{\vec{d}} f(1,1,1) = \nabla f(1,1,1) \cdot \vec{d}$$

$$= \frac{3 + 4 - 5}{5\sqrt{2}} = \frac{2}{5\sqrt{2}}$$

Maximum rate of change of ~~sa~~ scalar field :

$$D_{\vec{d}} f(\vec{a}) = \nabla f . \vec{d}$$

$$= |\nabla f| \, |\vec{d}| \cos\theta$$



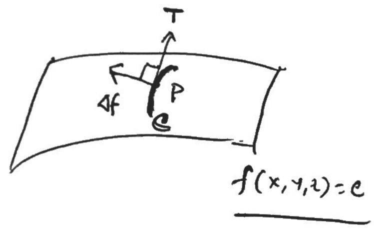
$$= |\nabla f| \cos\theta$$

Maximum rate of change, when $\theta = 0$, and the value $= \underline{|\nabla f|}$, and the direction is

$$\vec{d} = \frac{\nabla f}{|\nabla f|} .$$

∇f is perpendicular to the tangent plane to the level surface $f(x,y,z) = c$.

Let, C is a curve passing through $P(x_0, y_0, z_0)$ on the surface.



Any point on the curve C can be taken as : $(x(t), y(t), z(t)) = r(t)$

So,

$$f(r(t)) = c .$$

$$\therefore \frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt} = 0$$

$$\Rightarrow \nabla f . \frac{d\vec{r}}{dt} = 0 .$$

Now, $\left.\frac{d\vec{r}}{dt}\right|_{t=0}$ is a tangent vector to C at $P(x_0, y_0, z_0)$

Hence, $\nabla f(x_0, y_0, z_0)$ is orthogonal to tangent plane.

EXM:

$$f(x,y) = x^2+4y^2.$$

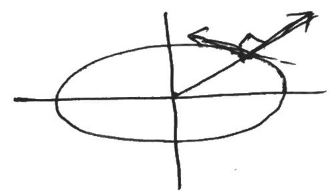
Compute ∇f, Show ∇f is normal to the level curve $f(x,y)=16$. Calculate the rate of change of f at $(1,1)$ in the direction $\vec{u}=(1,1)$.

i) $\nabla f = (f_x, f_y) = f_x\hat{i} + f_y\hat{j}$

$$= 2x\hat{i} + 8y\hat{j}.$$

ii) level Curve is : $x^2+4y^2=16$

$$\Rightarrow \frac{x^2}{16} + \frac{y^2}{4} = 1$$



$$\vec{r}(t) = (x(t), y(t))$$
$$= (4\cos t, 2\sin t)$$

$$\frac{d\vec{r}}{dt} = (-4\sin t, 2\cos t)$$

$$\nabla f = (8\cos t, 16\sin t)$$

$\therefore \nabla f \cdot \frac{d\vec{r}}{dt} = 0.$ So perpendicular to the tangent.

iii) $D_{\vec{d}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{d}$

$$= (2,8)\cdot(1,1)\cdot\frac{1}{\sqrt{1^2+1^2}}$$

$$= \frac{10}{\sqrt{2}}$$

HW. $f(x,y,z) = x^2+y^2-z^2.$

Compute the normal vector to the surface $f(x,y,z)=-7$ at $(1,1,3)$. Calculate $D_{\vec{d}} f(1,1,1)$ with $\vec{d}=(1,1,1)$.

Divergence of a vector field :-

Vector operator : $\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

Vector operator on scalar $\equiv$ gradient (vector).

Vector operator on vector $\equiv$ divergence (scalar) / Curl (vector)

\# Divergence of ~~f(x,y,z)~~ a vector $\vec{V} = V_1(x,y,z)\,\hat{i} + V_2(x,y,z)\,\hat{j} + V_3(x,y,z)\,\hat{k}$.

is ~~a~~ the scalar field :

$$\operatorname{div}(V) = \nabla\cdot\vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$$

\# $\nabla\cdot\vec{V} \neq \vec{V}\cdot\nabla$

$\uparrow$ Divergence $\qquad \uparrow$ scalar operator.

Exm :

$$\vec{F} = yz^2\,\hat{i} + xy\,\hat{j} + yz\,\hat{k}$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(yz^2) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(yz)$$

$$= \underline{x+y}$$

Curl :- Curl of a vector $\vec{v} = (v_1, v_2, v_3)$ is defined as the vector field

$$\operatorname{Curl}\vec{v} = \vec{\nabla}\times\vec{v}.$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Exm:

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & xy & yz \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(xy)\right)$$

$$- \hat{j}\left(\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial z}(yz^2)\right) + \hat{k}\left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(yz^2)\right)$$

$$= z\hat{i} + 2yz\hat{j} + (y - z^2)\hat{k}.$$

Properties :-

i) $\text{div}(\text{grad } f) = \nabla \cdot \nabla f = \Delta f = \nabla^2 f$

$f_{xx} + f_{yy} + f_{zz}$. Laplace operator

continuous 2nd order p.ds

ii) $\text{div}(\text{curl } \vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = 0$, for ~~differentiable~~ $\vec{F}$.

iii) $\text{Curl}(\text{grad } f) = \vec{0}$. for smooth f.

iv) $\text{div}(f\vec{v}) = f \,\text{div}\,\vec{v} + (\text{grad } f) \cdot \vec{v}$

$\left(\nabla \cdot (f\vec{v}) = f(\nabla \cdot \vec{v}) + (\nabla f) \cdot \vec{v}\right)$

v) $\text{curl}(f\vec{v}) = f \,\text{curl}(\vec{v}) + (\text{grad } f) \times \vec{v}$.

$\left(\nabla \times (f\vec{v}) = f(\nabla \times \vec{v}) + \nabla f \times \vec{v}\right.$

vi) $\text{curl}(\text{curl } \vec{v}) = \cancel{\nabla(\nabla \cdot \vec{v})}\ \text{grad}(\text{div } \vec{v}) - \cancel{\text{div}(\text{grad } \vec{v})}$ $\nabla^2 \vec{v}$

$\nabla \times (\nabla \times \vec{v}) = \nabla(\nabla \cdot \vec{v}) - \nabla^2 \vec{v}$

Vector Laplacian