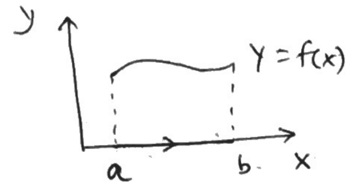
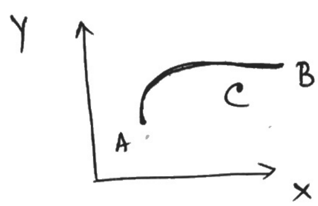
Line Integral :-

$$\int_a^b f(x)\,dx$$



integrating over an interval $[a,b]$.

$$\int_C f(x,y)\,ds \quad : \quad \text{integrating over a general curve } C$$



C : $\underline{(x(t), y(t))}$ parametrization of smooth curve

(i) $x^2+y^2=3^2$: $x = 3\sin t$, $y = 3\cos t$ | $x=t$, $y=\sqrt{3^2-t^2}$

(ii) $x=4y$, $z=3$: $y=t$, $x=4t$, $z=3$

$C: (4t, t, 3)$

(iii) Line joining $(0,0,0)$ & $(1,1,1)$: $\dfrac{x-0}{1-0} = \dfrac{y-0}{1-0} = \dfrac{z-0}{1-0} = t$

$C: (t,t,t)$

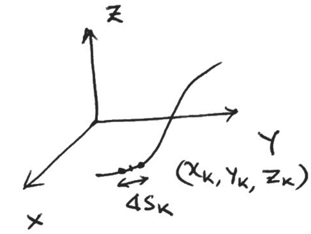
Trick: Convert the line integral ~~into~~ a simple definite integral by parametrization.

<u>Defn</u> :- Let, f is defined on a curve C, given by

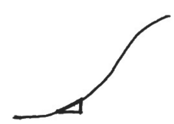
~~the~~ $\vec{r}(t) = (x(t), y(t), z(t))$ parametrically.

Then, the line integral of f over C is defined by ;

$$\int_C f(x,y,z)\, ds = \lim_{n \to \infty} \sum_{K=1}^{n} f(x_K, y_K, z_K)\, \Delta S_K$$



if the limit exists.



$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$, as Δs forms ~~the~~ a corner of the ~~parallelogram~~ Rectangular Prism

i.e. $$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

So,

$$\int_C f(x,y,z)\, ds = \int_a^b f(x(t), y(t), z(t))\, \frac{ds}{dt} \cdot dt$$

$$= \int_a^b f(x(t), y(t), z(t)) \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$$

Exm :

$$\int_C (2+x^2y)\,ds \qquad C \text{ is the upper half of } x^2+y^2=1$$

Parametrization: $x=\cos t,\ y=\sin t,\quad 0\le t\le \pi$

So, $$\int_C (2+x^2y)\,ds = \int_0^{\pi} (2+\cos^2 t\sin t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^{\pi} (2+\cos^2 t \sin t)\,dt$$

$$= 2\pi + \int_{-1}^{1} u^2\,du \qquad \begin{array}{r}\cos t = u\\ -\sin t\,dt = du\end{array}$$

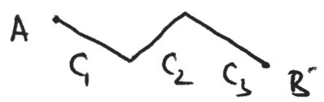
$$= 2\pi + \left[\frac{u^3}{3}\right]_{-1}^{1}$$

$$= 2\pi + \frac{2}{3}.$$

Additivity :-

If a piecewise smooth curve C is made by joining a finite no of smooth curves $C_1, \dots, C_n$ then,

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \dots + \int_{C_n} f\,ds.$$



Line integral of vector fields :-

Let, $\vec{F} = M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + P(x,y,z)\,\hat{k}$ be a vector field with continuous components on the curve $C = \cancel{(g(t), h(t), z(t))}\ \cancel{(g(t), h(t), k(t)),\ a\le t\le b}$.

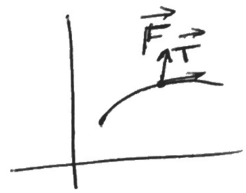
$$\vec{r}(t) = (x(t), y(t), z(t)).$$

Then, the line integral of $\vec{F}$ over C is defined by

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (M\,dx + N\,dy + P\,dz)$$

$$= \int_a^b \vec{F}(\vec{r}(t))\cdot \frac{d\vec{r}}{dt}\cdot dt$$

$$= \int_C \vec{F}\cdot \underbrace{\frac{d\vec{r}}{ds}}_{\vec{T}\,=\,\text{tangent vector.}}\, ds$$



~~Exm~~ %- <u>Geometrical Representation:</u> $\vec{F}\cdot\vec{T}$ is the scalar component of $\vec{F}$ in the direction of tangent $\vec{T}$.

$$\int_C \vec{F}\cdot T\,ds = \text{work done by } \vec{F} \text{ over the curve from } a \text{ to } b.$$

~~Exm~~ %- If the path of integration C is a closed curve, then we use the notation

$$\oint_C \text{ instead of } \int_C .$$

<u>Exm</u> :- $\int_C \vec{F}\cdot d\vec{r}$ where, $\vec{F} = ((y-x^2),\ z-y^2,\ x-z^2)$

$C: (t, t^2, t^3)$ from $(0,0,0)$ to $(1,1,1)$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r})\cdot \frac{d\vec{r}}{dt}\, dt \qquad \frac{d\vec{r}}{dt} = \hat{i} + 2t\hat{j} + 3t^2\hat{k}$$

$$= \int_0^1 \left[(t^3 - t^4)2t + 3t^2(t - t^6)\right] dt$$

$$= \frac{29}{60}.$$

Exm :- $\int_C (-y\,dx + z\,dy + 2x\,dz)$ $C:$ $r(t) = (\cos t, \sin t, t)$
$0 \le t \le 2\pi$

$$= \int_0^{2\pi} \left(-y\frac{dx}{dt} + z\frac{dy}{dt} + 2x\frac{dz}{dt}\right) dt$$

$$= \int_0^{2\pi} \left[(t\sin^2 t) + t\cos t + 2\cos t\right] dt$$

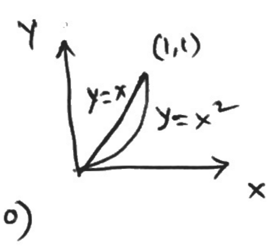
$$= \left[2\sin t\right]_0^{2\pi} + \left[t\sin t - \int \sin t\,dt\right]_0^{2\pi} + \frac{1}{2}\left(t - \frac{\sin 2t}{2}\right)\Big|_0^{2\pi}$$

$$= \cos t\Big]_0^{2\pi} + \pi = \pi.$$

Path Dependence :-

The line integral depends on the path along which the integral is taken.

Exm : $\vec{F} = (0, xy, 0)$



$C_1 :\ x = t,\ y = t,\ z = 0,\ \frac{d\vec{r}}{dt} = (1,1,0)$

$C_2 :\ x = t,\ y = t^2,\ z = 0,\ \frac{d\vec{r}}{dt} = (1, 2t, 0)$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^1 t^2\,dt = \frac{1}{3}.$$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^1 2t^4\,dt = \frac{2}{5}.$$

Path Independence :-

$\vec{F}$ is a vector field on (open connected) D. If for every pair of pts A & B in D, $\int_C \vec{F}\cdot d\vec{r}$ has the same value for all paths in D from A to B. Then, $\vec{F}$ is called <u>conservative</u> on D.

$\Rightarrow$ $\int_C \vec{F}\cdot d\vec{r}$ is path independent iff $\vec{F} = \text{grad}(f)$ for a potential function f.

Fundamental Theorem of Line integrals :-

Let, C be a smooth curve joining A & B, and parametrized by $\vec{r}(t)$. Let, f be a differentiable function with a continuous gradient vector $\vec{F} = \nabla f$ on D containing C. Then,

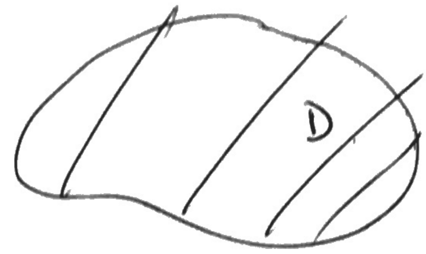
$$\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A)$$
$$= f(\vec{r}(b)) - f(\vec{r}(a)).$$

$\Rightarrow$ <u>$\int_C \vec{F}\cdot d\vec{r}$ is path independent</u> iff $\oint_C \vec{F}\cdot d\vec{r} = 0$ for every closed path C in D.

<u>EXM</u> :- Find the work done by the conservative field $\vec{F} = (yz, xz, xy) = \nabla f$, $f = xyz$ along the smooth curve C joining $A(-1,3,9)$ & $B(1,6-4)$

$\vec{F} = \nabla f \iff \vec{F}$ is conservative $\iff \oint_C \vec{F}\cdot d\vec{r} = 0$ for any loop C.

~~D: Simply connected domain.~~



Soln :- $f(x,y,z) = xyz$

$$\vec{C}: \frac{x+1}{1+1} = \frac{y-3}{6-3} = \frac{z-9}{-4-9} = t$$

$$\Rightarrow \vec{r} = (2t-1,\ 3t+3,\ -13t+9)$$

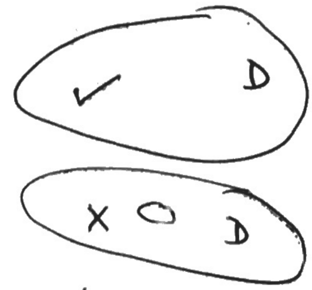
$$\int_C \vec{F}\cdot d\vec{r} = \int_A^B \nabla f \cdot d\vec{r}$$

$$= f(B) - f(A)$$

$$= xyz\Big|_{(1,6,-4)} - xyz\Big|_{(-1,3,9)}$$

$$= -24 + 27 = 3.$$

D: simply connected domain



Let, $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ is a vector field on simply connected region D, with F_i have cont. p.ds. Then, $\vec{F}$ is conservative iff $\text{curl}(\vec{F}) = \vec{0}$ or

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z},\quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x},\quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

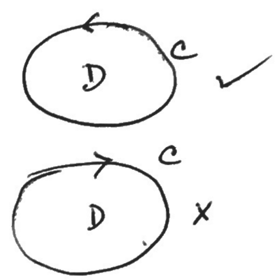
Green's Theorem (Relation between double integral & line integral)

Suppose the region D is bounded by a simple closed curve C. If D is always on the left of C, i.e. C travels in counterclockwise direction, C will be called positively oriented.

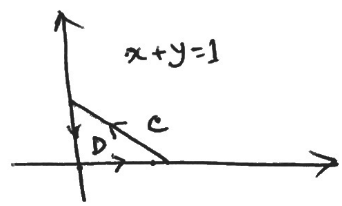
$\Rightarrow$ Let, C be a positively-oriented, simple closed piecewise smooth curve, bounding the region D. If $P(x,y)$ & $Q(x,y)$ are cont. and have cont. P.ds in D, then,

$$\oint_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

or
$$\oint_{\partial D} (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$



Exm:- $\int_C x^4\,dx + xy\,dy$, C is a triangular curve joining $(0,0)$ & $(1,0)$, $(1,0)$ & $(0,1)$ and $(0,1)$ & $(0,0)$



$P = x^4$, $Q = xy$.

$$\therefore \oint_C (x^4\,dx + xy\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D (y)\,dx\,dy$$

$$= \int_0^1 \left[\int_0^{1-x} y\,dy\right] dx = \frac{1}{2}\int_0^1 (1-x)^2\,dx = \frac{1}{6}.$$