Now, $\iint_D dx\,dy = A$.

$P = -\frac{1}{2}y$, $Q = \frac{1}{2}x$

then,

$$\frac{1}{2}\oint_C (x\,dy - y\,dx) = \iint_D dA = A$$

Surface Integrals :-

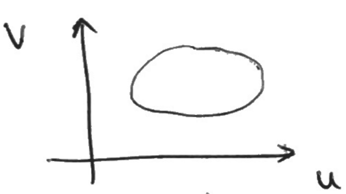
'Two-dimensional analog of line integrals'. The region of integration is a surface rather than a curve.

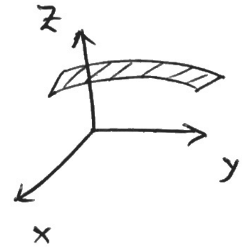
$z = f(x,y)$ or $F(x,y,z) = 0$ or

$\downarrow$

Parametrization :-

$$x = x(u,v),\quad y = y(u,v),\quad z = z(u,v).$$



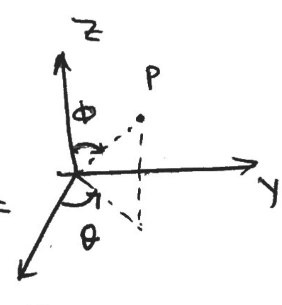


$$\vec{r}(u,v) = (x, y, z).$$

Sphere:

$$x^2+y^2+z^2 = a^2$$

$\cancel{x = a\sin\theta\cos\phi}$



$x = a\cos\theta\cos\phi$, $y = a\sin\theta\cos\phi$

$z = a\sin\phi$.

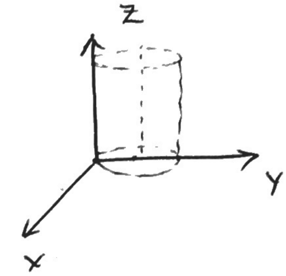
$\theta \in [0, 2\pi]$, $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$x = a\sin\phi\cos\theta, \quad y = a\sin\phi\sin\theta, \quad z = a\cos\phi$

$0 \le \phi \le \pi, \qquad 0 \le \theta \le 2\pi$

Cylinder ~~Cone~~ :-

$x^2 + (y-3)^2 = 9, \quad 0 \le z \le 5$



Let, $x = r\cos\theta$

$y = r\sin\theta, \qquad z = z$

$\Rightarrow \quad r^2 - 6r\sin\theta = 0$

$\Rightarrow \quad r = 6\sin\theta \qquad 0 \le \theta \le \pi$

Cylindrical co-ordinates

$x = 3\sin 2\theta$

$y = 6\sin^2\theta, \qquad z = z$

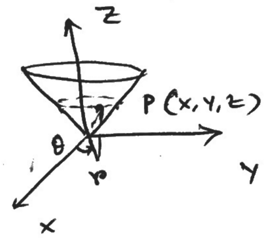
$r(\theta, z) = (3\sin 2\theta, 6\sin^2\theta, z) \qquad 0 \le \theta \le \pi, \quad 0 \le z \le 5$

Cone :-

$z = \sqrt{x^2+y^2} \qquad 0 \le z \le 1$

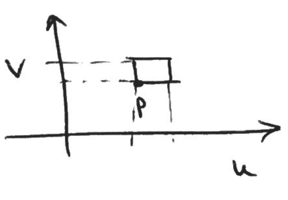
$x = r\cos\theta, \quad y = r\sin\theta$

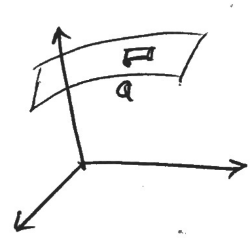
$r = \sqrt{x^2+y^2} = z$



$0 \le r \le 1, \qquad 0 \le \theta \le 2\pi$

$\vec{r}(r, \theta) = (r\cos\theta, r\sin\theta, r)$





$\vec{r}(u, v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$

$\vec{r}_u$ & $\vec{r}_v$ give tangent to curves on the surface.

So, $\vec{r}_u \times \vec{r}_v$ is normal to the surface.

If the surface is given by $F(x,y,z) = c$; then,

$$\vec{r}_u \times \vec{r}_v = \frac{\nabla F}{\nabla F \cdot \vec{p}}$$

, $\vec{p}$ is the unit normal to the plane region R on which the surface is considered.

Scalar fn :-

Defn :-

i) If S is given parametrically.

$\vec{r}(u,v) = (x(u,v),\ y(u,v),\ z(u,v))$ and $G(x,y,z)$ is defined ~~...~~ and continuous on S,

then,

$$\iint_S G(x,y,z)\, ds = \iint_R G(x(u,v), y(u,v), z(u,v))\, |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

$$ds \simeq |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

Line integral $\leftrightarrow$ definite integral

Surface " $\leftrightarrow$ double integral

ii) Implicit : $F(x,y,z) = c$, then,

$$\iint_S G(x,y,z)\, ds = \iint_R G(x,y,z) \frac{|\nabla F|}{|\nabla F \cdot \vec{p}|}\, dA$$

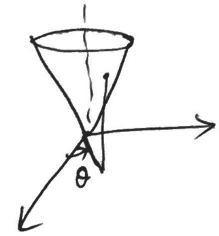
$\vec{p}$ is a unit normal vector to R.

iii) Explicit : $z = f(x,y)$

$$\iint_S G(x,y,z)\, ds = \iint_R G(x,y,f(x,y)) \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy.$$

EXM :-

$$\iint_S x^2\, dS \qquad S: \text{ cone } \; z = \sqrt{x^2+y^2}, \quad 0 \le z \le 1$$



$r^2 = x^2 + y^2 = z^2$

& $x = r\cos\theta$

$y = r\sin\theta$

$0 \le r \le 1, \qquad 0 \le \theta \le 2\pi.$

$\vec{r}(r,\theta) = (r\cos\theta,\ r\sin\theta,\ r)$

$\vec{r}_r = (\cos\theta,\ \sin\theta,\ 1) \qquad \vec{r}_\theta = (-r\sin\theta,\ r\cos\theta,\ 0)$

du dv.

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-r\cos\theta,\ -r\sin\theta,\ r)$$

$\therefore\ |\vec{r}_r \times \vec{r}_\theta| = \sqrt{2}\, r.$

So,
$$\iint_S x^2\, ds = \int_0^{1}\int_0^{2\pi} r^2\cos^2\theta\, \sqrt{2}\, r\, dr\, d\theta$$

$$= \sqrt{2}\ \frac{1}{4}\cdot\frac{1}{2}\cdot\left(2\pi + \left[\frac{\sin 2\theta}{2}\right]_0^{2\pi}\right)$$

$$= \frac{\sqrt{2}}{8}(2\pi) = \frac{\pi\sqrt{2}}{4}.$$

EXM :-

$$\iint_S y\, ds \qquad S: \ z = x + y^2, \quad 0 \le x \le 1, \quad 0 \le y \le 2$$

dx dy.

$z_x = 1, \quad z_y = 2y$

$$\iint_S y\, ds = \iint_D y\sqrt{1+1+4y^2}\, dx\, dy$$

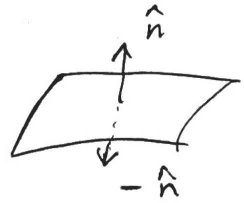
$$= \sqrt{2}\int_0^1\left[\int_0^2 y\sqrt{1+2y^2}\, dy\right] dx = \frac{13\sqrt{2}}{3}.$$

Vector field :-

A surface S is orientable if it has two sides.

Exm of Non-orientable surface: Möbius strip.

Normal ~~vec~~ vectors : $\hat{n}$ and $-\hat{n}$.



i> $$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

ii> $$\hat{n} = \frac{-f_x \hat{i} - f_y \hat{j} + \hat{k}}{\sqrt{1 + f_x^2 + f_y^2}} \quad : z = f(x,y)$$

$\Rightarrow$ For a closed region, outward direction is the +ve orientation.

Defn : Let, $\vec{F}$ is a continuous vector field on an oriented surface S with unit normal $\hat{n}$, then the surface ~~intg~~ integral of $\vec{F}$ over S is

$$\iint_S \vec{F} \cdot \hat{n} \, ds .$$

This is called the flux of $\vec{F}$ across S.

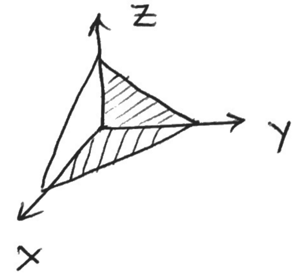
So,

$$\iint_S \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} ds = \iint_R \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v| \, dA$$

$$= \iint_R \vec{F} \cdot \vec{r}_u \times \vec{r}_v \, dA$$

Exm :- $\iint_S \vec{F}\cdot d\vec{S}$ $\quad \vec{F} = (x^2, 0, 3y^2)$

S: $x+y+z=1$ in 1st octant



$X = u, \quad Z = v$

$Z = 1-u-v$

$\vec{r}(u,v) = (u, v, 1-u-v)$

$\vec{r}_u = (+1, 0, -1)$

$\vec{r}_v = (0, 1, -1)$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$

$= (1, 1, 1)$

$\iint_S \vec{F}\cdot\hat{n}\, dS =$

$\cancel{= \int_0^1\int_{v=0}^{1-u} \left((1-u-v)^2 + 3v^2\right) du\, dv}$

$\cancel{\int_0^1\int_0^{1-u} 1+u^2+4v^2-2u+2uv}$

$= \int_0^1\int_{v=0}^{1-u} (u^2+3v^2)\, dv\, du = \int_0^1 \left[u^2(1-u) + (1-u)^3\right] du$

$= \int_0^1 (1-u)(u^2+1+u^2-2u)\, du$

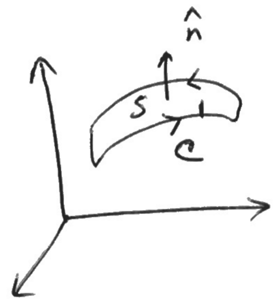
$= \int_0^1 (1-u)(2u^2-2u+1)\, du$

$= \frac{2}{3} - \cancel{1} + \cancel{1} - \frac{2}{4} + \frac{2}{3} - \frac{1}{2}$

$= \frac{4}{3} - 1 = \frac{1}{3}.$

Stokes's theorem :-

+ve orientation of C, boundary curve of S.



If Head up in $\hat{n}$ direction, S will be always on the left.

Let, S be an oriented piecewise smooth surface, that is bounded by a simple, closed piecewise smooth curve C with positive orientation. A vector field $\vec{F}$ has cont. components on S. Then,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds$$

line integral ⟺ surface integral

$$\boxed{\text{Stokes' + xy-plane = Green's}}$$

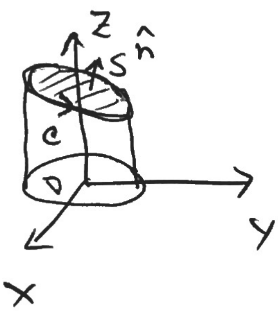
Exm :-

$$\int_C \vec{F} \cdot d\vec{r} \qquad \vec{F} = (-y^2, x, z^2)$$

C : intersection of $y+z=2$ & $x^2+y^2=1$

By stokes'

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, ds$$



$$\nabla \times \vec{F} = (1+2y)\hat{k}$$

$$\hat{n} = \frac{-g_x\hat{i} - g_y\hat{j} + \hat{k}}{\sqrt{1+g_x^2+g_y^2}}$$

$y+z=2$

$\Rightarrow z = 2-y = g(x,y)$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_D \nabla\times\vec{F}\cdot\hat{n}\sqrt{1+g_x^2+g_y^2}\,dA \qquad D: \underline{x^2+y^2\le 1}$$

$$= \iint_D (1+2y)\,\hat{k}\cdot\hat{k}\,dA$$

$$= \iint_D (1+2y)\,dA \qquad \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}$$

$$= \int_0^1\int_0^{2\pi}(1+2r\sin\theta)\,r\,dr\,d\theta \qquad \begin{array}{l} 0\le\theta\le 2\pi \\ 0\le r\le 1 \end{array}$$

$$= \pi.$$

Gauss Divergence Theorem :-

Let, D be a closed, bounded region whose boundary is a piecewise smooth surface S that is oriented outward. Let, $\vec{F}$ be a vector field whose components have cont. 1st order P.ds. on D. Then,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_D \nabla\cdot\vec{F}\,dv.$$

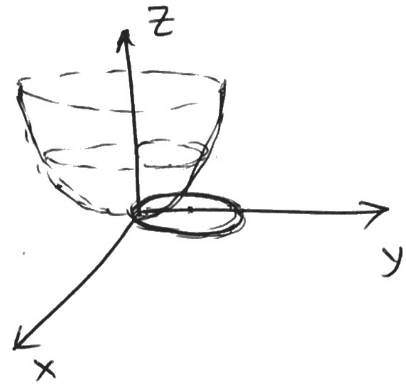
where $\hat{n}$ is the outer unit normal to S.

Exm :-

$$\iint_S \vec{v}\cdot\hat{n}\,ds \qquad \vec{v} = (x^2 z,\ y,\ -xz^2)$$

$$S:\ z = x^2+y^2,\ z = 4y$$

Intersection



$$x^2+y^2-4y=0$$

$$\Rightarrow x^2+(y-2)^2 = 4$$

Projection of S on xy plane

$$x^2+(y-2)^2=4$$

D.T

$$\iint_S \vec{v}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{v}\,dV$$

$$= \iiint_V (\cancel{2xz}+1-\cancel{2xz})\,dV$$

$$= \int_0^4 \int_{x=-\sqrt{4y-y^2}}^{\sqrt{4y-y^2}} \int_{z=x^2+y^2}^{4y} dx\,dy\,dz$$

$$= \int_0^4 \int_{-\sqrt{4y-y^2}}^{\sqrt{4y-y^2}} (4y-x^2-y^2)\,dx\,dy$$

$$= \int_0^4 \frac{4}{3}(4y-y^2)^{3/2}\,dy$$

$$= \frac{4}{3}\int_0^4 \left[4-(y-2)^2\right]^{3/2} dy \qquad y-2 = 2\sin t$$

$$= 8\pi.$$