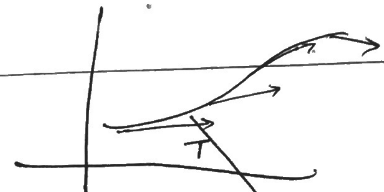
Curvature :-

A "curvy"

B "less curvy"

C

Curvature is a local concept of a curve of measuring "How curvy" the figure is.



The rate of change of $\vec{T}$ per unit arc length.

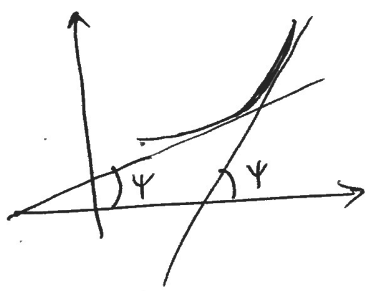
$$\kappa \text{ (Kappa)} = \left|\frac{d\vec{T}}{ds}\right|.$$

$\vec{T}$: unit tangent vector at a point P.

If the curve is parametrized in t, then,

$$\left|\frac{d\vec{T}}{ds}\right| = \frac{d\vec{T}}{}$$

Measure how quickly the angle ψ changes as we move along the curve.



$$\kappa \text{ (Kappa)} = \left|\frac{d\psi}{ds}\right|.$$

If $y = f(x)$, then,

$$\frac{d\psi}{ds} = \frac{d\psi}{dx} \cdot \frac{dx}{ds}$$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \left(1+[f'(x)]^2\right)^{1/2}$$

And, $\quad \frac{dy}{dx} = \tan\psi$.

$$\Rightarrow \frac{d^2y}{dx^2} = \sec^2\psi \, \frac{d\psi}{dx}$$

$$\therefore \frac{d^2y}{dx^2} = (1+\tan^2\psi)\,\frac{d\psi}{dx}$$

$$\therefore \frac{d\psi}{dx} = \frac{y''}{(1+y'^2)}$$

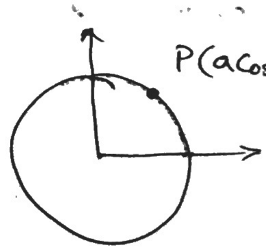
$$\therefore K = \left|\frac{d\psi}{dx} \Big/ \frac{ds}{dx}\right| = \left|\frac{y''}{(1+y'^2)^{3/2}}\right|$$

\# If $x = g(t)$, $y = h(t)$. parametric curve.

then, $$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \dot{y}/\dot{x}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$$

$$\therefore K = \left|\frac{y''}{(1+y'^2)^{3/2}}\right| = \left|\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2+\dot{y}^2)^{3/2}}\right|$$



Exm : $\quad x = a\cos t, \quad y = a\sin t$

$$K = \left|\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2+\dot{y}^2)^{3/2}}\right|$$

$$= \left|\frac{a^2}{a^3}\right| = \frac{1}{a}$$

Radius of curvature at P, $\rho = \frac{1}{K}$.

The curvature of a curve $\vec{r}(t)$

is $K(t) = \dfrac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$

$$\left\{ \begin{array}{l} \frac{ds}{dt} = |r'(t)| \\ T(t) = \frac{r'(t)}{|r'(t)|} \\ K(t) = \left|\frac{dT}{ds}\right| \end{array} \right.$$

or, $K(t) = \dfrac{1}{|v|}\left|\dfrac{d\vec{T}}{dt}\right|$

with $\underline{v = r'(t)}$ & $\vec{T} = \dfrac{\vec{r}\,'(t)}{|r'(t)|}$

Exm: $\vec{r}(t) = (t, t^2, t^3)$

$\cancel{\vec{v}(t)}$ $\vec{r}\,'(t) = (1, 2t, 3t^2)$, $\vec{r}\,''(t) = (0, 2, 6t)$

$\cancel{|\vec{v}| = \sqrt{1+4t^2+9t^4}}$

$\cancel{\vec{T} = \frac{\vec{v}}{|\vec{v}|}} =$ $\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix}$

$= (6t^2, -6t, 2)$

& $\vec{r}\,'(t) = \sqrt{1+4t^2+9t^4}$

at, a sint)

$\therefore$ $K(t) = \dfrac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} = \dfrac{\sqrt{36t^4 + 36t^2 + 4}}{(1+4t^2+9t^4)^{3/2}}$

$\therefore$ $K(0) = 2$.

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$$y = x^2 = f(x).$$

$$f'(x) = 2x \quad , \quad f''(x) = 2$$

$$K = \frac{|f''(x)|}{\left(1+f'(x)^2\right)^{3/2}} = \frac{2}{\left(1+4x^2\right)^{3/2}}$$

$$\therefore\ K\Big|_{(0,0)} = 2, \quad K\Big|_{(1,1)} = \frac{2}{5^{3/2}}, \quad K\Big|_{2,4} = \frac{2}{(17)^{3/2}}$$

Exm:

$$x = a(\cos\theta + \theta\sin\theta), \quad y = a(\sin\theta - \theta\cos\theta)$$

$$K = \left|\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}}\right| \qquad \begin{aligned}\dot{x} &= a\theta\cos\theta \\ \dot{y} &= \cancel{a(\cos\theta} a\theta\sin\theta.\end{aligned}$$

$$= \left|\frac{a^2(-\cancel{\sin\theta} + \theta\cos\theta + \cancel{\sin\theta})(\sin\theta + \theta\cos\theta) - a^2\theta\sin\theta(\cos\theta - \theta\sin\theta)}{a^3\theta^3}\right|$$

$$= \left|\frac{a^2\theta\cos\theta(\cancel{\sin\theta} + \theta\cos\theta) - a^2\theta\sin\theta(\cancel{\cos\theta} - \theta\sin\theta)}{a^3\theta^3}\right|$$

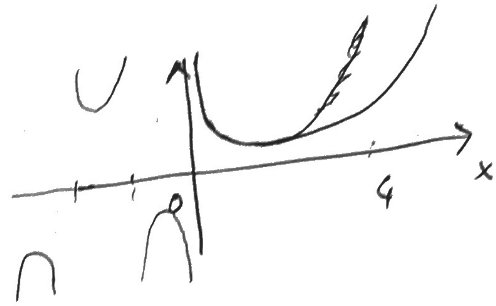
$$= \left|\frac{a^2\theta^2}{a^3\theta^3}\right| = \frac{1}{a\theta}.$$

$$\therefore\ \underline{K \propto \frac{1}{\theta}} \qquad \text{or } \rho = \frac{1}{K} = a\theta$$

$$\therefore\ \underline{\rho \propto \theta}$$

Beta & Gamma Functions :-

$$\Gamma(\alpha) := \int_0^\infty e^{-t}\, t^{\alpha-1}\, dt \quad , \quad \alpha > 0$$



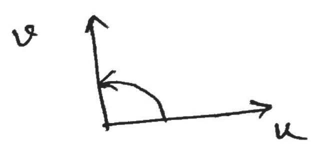
Properties:

1. $$\Gamma(1) = \int_0^\infty e^{-t}\, dt = -e^{-t}\Big]_0^\infty = 1 .$$

2. $$\Gamma(\alpha+1) = \int_0^\infty e^{-t}\, t^{\alpha}\, dt$$
$$= -t^{\alpha} e^{-t}\Big]_0^\infty + \int_0^\infty \alpha\, t^{\alpha-1} e^{-t}\, dt$$
$$= \alpha\, \Gamma(\alpha) .$$

3. $$\Gamma(2) = 1 \cdot \Gamma(1) = 1 , \qquad \Gamma(3) = 2 \cdot \Gamma(2) = 2!$$
$$\ldots \qquad \Gamma(n+1) = n! \, , \quad n = 0, 1, \ldots$$

4. $$\Gamma(\tfrac{1}{2}) = \int_0^\infty e^{-t}\, t^{-1/2}\, dt \qquad\qquad t = u^2, \quad dt = 2u\, du$$
$$= 2\int_0^\infty e^{-u^2}\, du$$



$$\left[\Gamma(\tfrac{1}{2})\right]^2 = 4 \int_0^\infty e^{-u^2}\, du \int_0^\infty e^{-v^2}\, dv$$
$$= 4 \int_0^\infty\!\!\int_0^\infty e^{-u^2} \cdot e^{-v^2}\, du\, dv \qquad\qquad u = r\cos\theta, \quad v = r\sin\theta, \quad J(r,\theta) = r$$
$$= 4 \int_{r=0}^\infty \int_{\theta=0}^{\pi/2} e^{-r^2}\, r\, dr\, d\theta \qquad\qquad r^2 = z, \quad 2r\,dr = dz$$
$$= 2 \cdot \frac{\pi}{2} \cdot \int_0^\infty e^{-z}\, dz$$
$$= \pi$$

$$\Rightarrow \quad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}.$$

5. $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$

$$\alpha = -\tfrac{1}{2} \Rightarrow \Gamma\left(-\tfrac{1}{2}\right) = \frac{\Gamma(1/2)}{-1/2} = -2\sqrt{\pi}.$$

Extend for ~~non~~ negative non-integers.

6. Complement property :

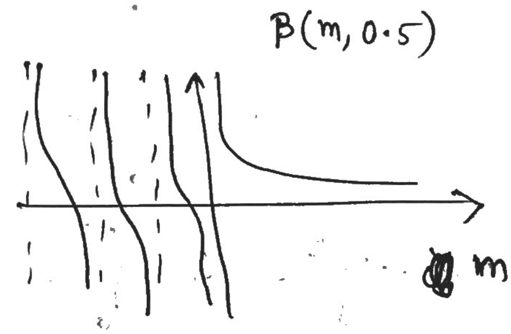
$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \quad x \notin \mathbb{Z}$$

7. $\Gamma(x)$ is differentiable for $x>0$,

$$\Gamma'(x) = \int_0^\infty e^{-t} \ln(t)\, t^{x-1}\, dt$$

Beta Functions :-

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \quad m>0,\ n>0$$



Properties :-

1. $$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

2. $B(m,n) = B(n,m)$ (Symmetry)

3. $$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta \qquad x = \sin^2\theta$$

4. $$B(m,n) = \int_0^\infty \frac{u^{m-1}}{(1+u)^{m+n}}\,du, \quad u = \frac{x}{1-x}$$