ODE :-

An ODE contains one or several derivatives of an unknown function $y(x)$:

$$F(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0$$

Order :- The order of the highest order derivative in the ODE.

$$\frac{d^3y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = \cancel{f(x)}\, \sin(x). \quad \cdots (*)$$

order : 3.

Degree :- The degree or power of the highest order derivatives, after the eqn is made free of radicals and fractions in its derivatives.

$(*)$ degree : 1.

$$\left(\frac{d^2y}{dx^2}\right)^{3/2} + y = x$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^3 = (x-y)^2$$

order : 2 degree = 3.

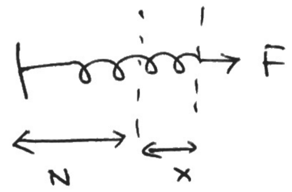
Exm :- The rate of growth of population is proportional to the population size :

$$\frac{dP}{dt} \propto P$$

$$\Rightarrow \frac{dP}{dt} = KP$$

2. Hooke's law:

$$m \frac{d^2x}{dt^2} = -Kx$$



Force $\propto$ stretched length.

~~Linear ODE~~

First order ODE :-

$$F(x, y, y') = 0 \quad : \cancel{\text{Explicit}} \text{ Implicit}$$

or $$y' = \frac{dy}{dx} = f(x, y) \quad : f \text{ may be linear or non-linear in } y.$$

Exm:

$$y' = x \Rightarrow \frac{dy}{dx} = x \qquad \underline{\text{Explicit}}.$$

$$\Rightarrow y = \frac{x^2}{2} + c, \quad c = \text{Integration const.}$$

Suppose, $y(0) = 0$, $\therefore c = 0$

$$\underline{y = x^2/2}$$

$\rightarrow$ Initial condition.

Initial Value Problems (IVP)

$$\underline{y' = f(x, y), \; y(x_0) = y_0} \quad \cdots (1)$$

Solution of ODE :- A function $y = \phi(x)$ is a soln of (1) if

i) $\phi \in C^1$

ii) $\phi'(x) = f(x, \phi(x))$, $\phi(x_0) = y_0$

Is it always possible to get a soln of IVP?

$$|y'(x)| + |y(x)| = -1, \quad y(0) = 0 \;/\; y'^2 + y^2 + 1 = 0, \; y(0) = 1$$

Linear ODE : If F is linear in y and its derivatives, it is called a linear ODE.

$$y' + xy = x^2$$

If the eqn is not linear, it is called non-linear.

$$yy' = x.$$

Existence & Uniqueness :-

$$xy' = y - 1, \quad y(0) = 1$$

$\rightarrow$ $y(x) = 1 + \alpha x$, α is any constant

Not unique.

Theorem :- Let, f is continuous in x and y in D. Let, $\frac{\partial f}{\partial y}$ is a continuous function in x & y in D. $(x_0, y_0) \in D$.

Then, $\exists$ a unique soln ϕ of

$$y' = f(x, y), \quad y(x_0) = y_0$$

on some interval $|x - x_0| \le h$ with $\phi(x_0) = y_0$.

(1) $\Rightarrow$ n^{th} order eqn have n arbitrary constants in its general solution.

Solution Methods :

1. Separable ODE.

$$g(y)y' = f(x).$$

$$\Rightarrow \int g(y)\,dy = \int f(x)\,dx + c$$

Exm : $y' = 1+y^2$

$$\Rightarrow \int \frac{dy}{1+y^2} = \int dx + c$$

$$\Rightarrow \tan^{-1} y = x + c \Rightarrow y = \tan(x+c)$$

$y' = f(ax+by+c)$

$ax+by+c = u$

$$\Rightarrow \frac{du}{dx} = a + bf(u)$$

H.W : $y' = -2xy,\ y(0) = 2.$

2. Exact ODE, Integrating factors :-

If $u(x,y) = c$, and $u(x,y)$ has cont. P.ds, then,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

i.e. $$\frac{dy}{dx} = - u_x/u_y \quad \cdots (*)$$

Going backward we can solve (*) to $u(x,y) = c$.

\# $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be exact if it can be written as

$$du = u_x\,dx + u_y\,dy = 0$$

i.e. $u_x = M,\quad u_y = N.$

Now, if M & N have cont. p-ds., then,

$$\frac{\partial M}{\partial y} = u_{xy} = u_{yx} = \frac{\partial N}{\partial x}.$$

— The NASC for exact ODE.

Exm:- $(y+x^3)\,dx + (x-4y^3)\,dy = 0$

$$\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}.$$

So, the ODE is exact.

Now, $du = u_x\,dx + u_y\,dy \equiv M dx + N dy$

with $u_x = M = y + x^3$

$\Rightarrow u(x,y) = \int_x u_x\,dx = xy + \frac{x^4}{4} + g(y)$

$\therefore u_y = x + g'(y) = N = x - 4y^3$

$\Rightarrow g'(y) = -4y^3$

$\therefore g(y) = -y^4 + c$.

$\therefore u(x,y) = xy + \frac{x^4}{4} - y^4 = K$ (solⁿ.)

Integrating Factors:-

Suppose $M dx + N dy = 0$ is not exact. But,

$\mu(x,y) M\,dx + \mu(x,y) N dy$ is exact, then, μ is called an integrating factor

Different cases :-

1. Linear first order ODE: $y' = f(x, y)$

$$\frac{dy}{dx} + P(x)y = R(x).$$

$$I.F. = e^{\int P(x)dx}.$$

Exm:- $x^2y' + xy = 1, \quad y(1) = 2$

$$\Rightarrow y' + \frac{1}{x}y = \frac{1}{x^2}$$

$$I.F = e^{\int \frac{1}{x}dx} = e^{\ln(x)} = x.$$

$$\therefore \int d(xy) = \int \frac{1}{x}dx + c$$

$$\Rightarrow xy = \ln(x) + c$$

$$\therefore \cancel{y = \frac{\ln(x)}{x} + \frac{c}{x}}.$$

$y(1) = 2 \Rightarrow 2 = c \Rightarrow \underline{xy = \ln(x) + 2}$.

Exm: $y' + y\tan x = \sin(2x), \quad y(0) = 1$

$$I.F. = e^{\int \tan x\,dx} = e^{\ln|\sec x|} = |\sec x|$$

$$\therefore \int d(y\sec x) = \int 2\sin x\,dx + c$$

$$\Rightarrow y\sec x = -2\cos x + c$$

$y(0) = 1 \Rightarrow 1 = -2 + c \Rightarrow c = 3.$

$$\therefore \underline{y = -2\cos^2 x + 3\cos x}$$

2. Non-linear 1 :-

$$M\,dx + N\,dy = 0 \quad \cdots (4)$$

If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$, then, I.F. $= e^{\int f(x)\,dx}$.

EXM: $(5x^3 + 12x^2 + 6y^2)\,dx + 6xy\,dy = 0$

$$\frac{\partial M}{\partial y} = 12y, \quad \frac{\partial N}{\partial x} = 6y$$

$$\therefore \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{6y}{6xy} = \frac{1}{x} = f(x).$$

$$\therefore \text{I.F.} = e^{\int \frac{1}{x}dx} = x.$$

$$(5x^4 + 12x^3 + 6y^2x)\,dx + 6x^2y\,dy = 0$$

$$\Rightarrow (5x^4 + 12x^3)\,dx + 6xy\,d(xy) = 0$$

$$\therefore x^5 + 3x^4 + 3x^2y^2 = c$$

3. Non-linear 2 :-

If $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y)$, then, I.F. $= e^{-\int g(y)\,dy}$.

EXM :- $(e^{x+y} + ye^y)\,dx + (xe^y - 1)\,dy = 0$, $y(0) = -1$

$$\frac{\partial M}{\partial y} = e^{x+y} + (y+1)e^y, \quad \frac{\partial N}{\partial x} = e^y.$$

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{e^{x+y} + ye^y}{e^{x+y} + ye^y} = 1$$

$$\therefore \text{I.F.} = e^{-\int dy} = e^{-y}.$$

$$(e^x + y)dx + (x - e^{-y})\,dy = 0$$

$$\Rightarrow \quad e^x dx + d(xy) - e^{-y} dy = 0$$

$$\Rightarrow \quad e^x + xy + e^{-y} = c \qquad y(0) = -1$$

$$\Rightarrow \quad 1 + e = c$$

$$\therefore \quad e^x + xy + e^{-y} = 1 + e$$

4. Some readymade forms :-

i) $$\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$$

ii) $$\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$$

iii) $$x\,dy + y\,dx = d(xy)$$

iv) $$\frac{x\,dy - y\,dx}{x^2+y^2} = \frac{\frac{x\,dy - y\,dx}{x^2}}{1 + (y/x)^2} = \frac{d(y/x)}{1+(y/x)^2} = d\left(\tan^{-1}(y/x)\right)$$

v) $$\frac{x\,dx + y\,dy}{x^2+y^2} = \frac{1}{2} d\left(\ln(x^2+y^2)\right)$$

H.W. $$y\,dx - x\,dy + e^{1/x}\,dx = 0$$