Bernoulli Equation :-

$$\frac{dy}{dx} + P(x)y = R(x)y^n, \quad n \in \mathbb{R}.$$

~~Let,~~ $\cancel{v = y^{1-n} = \frac{1}{y^{n-1}}}$

Multiply both sides by y^{-n}.

$$y^{-n}\frac{dy}{dx} + P(x)\frac{1}{y^{n-1}} = R(x). \quad \cdots (1)$$

Let, $\cancel{v(x)}$ $v = \frac{1}{y^{n-1}}$

$$\therefore \frac{dv}{dx} = \frac{(1-n)}{y^n}\frac{dy}{dx}.$$

So, (1) $\Rightarrow$

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)R(x).$$

$$\frac{dv}{dx} + P_1(x)v = R_1(x) \quad : \text{ Standard Linear ODE.}$$

Exm:

$$\frac{dy}{dx} - y = y^2(\sin x + \cos x)$$

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = (\sin x + \cos x)$$

Let, $\frac{1}{y} = v \quad \therefore -\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$

$$\Rightarrow + \frac{dv}{dx} + v = -(\sin(x) + \cos x)$$

$$\text{I.F.} = e^{+\int dx} = e^{+x}$$

Hence, $\int d(ve^{+x}) = -\int e^{+x}\sin x\,dx - \int e^{+x}\cos x\,dx + c$

$$\therefore ve^{-x} = \mp e^{+x}\sin x + \int \cos x\, \cancel{e^{-x}}\,dx - \int e^{-x}\cancel{\cos}x\,dx + c$$

$$\therefore\ v e^{+x} = -e^{+x}\sin x + c$$

$$\therefore\ \underline{1 = y\left(\sin x + c e^{-x}\right)}.$$

H.W.

$$\frac{dy}{dx} + y = x y^3 \quad , \quad x y' - 2y = 4x^3 y^{1/2}$$

2nd Order Linear ODE :-

$$F(x, y, y', y'') = 0$$

$$y'' + P(x) y' + Q(x) y = R(x) \quad : \text{standard form} \quad \cdots (*)$$

If $R(x) = 0$, $(*)$ is called homogeneous, else non-homogeneous.

General Higher order ~~ODE~~ linear ODE :-

$$a_0(x) y^{(n)} + a_1(x) y^{(n-1)} + \cdots + a_n(x) y = f(x) \quad \cdot (*)$$

If $y_1, y_2, y_3 \ldots, y_n$ are n solutions of the homogeneous eqn. then, $c_1 y_1 + \cdots + c_n y_n$ is also a soln. of the same. $(**)$

(Superposition).

If $y_1, y_2 \ldots, y_n$ are linearly independent, then, they are called the fundamental soln. of $(*)$ with $\underline{f = 0}$.

\# $y_1(x), \ldots, y_n(x)$ are lin. indep. if

$$c_1 y_1(x) + \cdots + c_n y_n(x) = 0$$

$$\Rightarrow \quad c_1 = 0 = c_2 = \cdots = c_n.$$

Wronskian of m functions $g_1(x), \dots, g_m(x)$ is

$$W(g_1, \dots, g_m) = \begin{vmatrix} g_1(x) & g_2(x) & \dots & g_m(x) \\ g_1'(x) & \dots & & g_m'(x) \\ \vdots & & & \\ g_1^{(m-1)}(x) & \dots & & g_m^{(m-1)}(x) \end{vmatrix}$$

$\Rightarrow$ $g_1(x), \dots, g_m(x)$ are lin. indep. iff

$W(g_1, \dots, g_m) \neq 0 \quad \forall x$.

(**) Not true for non-homogeneous or non-lin. eqn

$y'' + y = 1$, $\quad y_1(x) = 1 + \cos x$, $\quad y_2(x) = 1 + \sin x$ are soln.

But, not $2y_1(x)$ or $5y_2(x)$.

\# $yy'' - xy' = 0$ has soln $y_1 = x^2$, $y_2 = 1$. But, $-x^2$ is not a soln.

IVP: $y'' + P(x)y' + Q(x)y = R(x)$, $y(x_0) = y_0$, $y'(x_0) = z_0$.

BVP: $y'' + P(x)y' + Q(x)y = R(x)$, $y(x_0) = y_0$, $y(x_1) = y_1$.

General soln :- If $y_1(x)$ & $y_2(x)$ are two lin-indep. soln of $y'' + P(x)y' + Q(x)y = 0$, then,

$y(x) = c_1y_1 + c_2y_2$ is a general soln of the ODE.

Exm:-

e^x, e^{2x}, e^{3x} are fundamental soln of

$$y''' - 6y'' + 11y' - 6y = 0$$

$$y' = 2e^{2x},\ y'' = 4e^{2x},\ y''' = 8e^{2x}$$

$$y''' - 6y'' + 11y' - 6y = 8e^{2x} - 24e^{2x} + 22e^{2x} - 6e^{2x}$$

$$= 0.$$

Also,

$$W(e^x, e^{2x}, e^{3x}) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$= 6e^x \cdot e^{5x} - 6e^{2x} \cdot e^{4x} + 2e^{3x} \cdot e^{3x}$$

$$= 2e^{6x} \neq 0.$$

So, they are lin indep. So, fundamental soln.

Linear Homogeneous Equation with Constant Co-efficient :-

$$ay'' + by' + cy = 0 \quad \cdots (1)$$

If $a = 0$, & $b \neq 0$. $y' = -\frac{c}{b} y$

$$\Rightarrow \int \frac{dy}{y} = -\frac{c}{b}\int dx + K$$

$$\Rightarrow \ln y = -\frac{c}{b}x + K.$$

$$\therefore y = Ae^{-c/b\, x}$$

The soln is of the form $y = Ae^{mx}$

So, it is natural to try $y = e^{mx}$ for (1).

Then, $y' = me^{mx}$

& $y'' = m^2 e^{mx}$

So, $(am^2 + bm + c) e^{mx} = 0$

$\Rightarrow \quad am^2 + bm + c = 0$

This is called the auxiliary equation.

Now, the roots are

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = m_1, m_2$$

We have ~~to~~ three cases:

Case - I: Two distinct real roots $(m_1 \neq m_2)$

Then, the corresponding general soln is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Exm: $y'' + y' - 2y = 0$

$m^2 + m - 2 = 0$

$\Rightarrow m = -2, 1, \quad y = c_1 e^{x} + c_2 e^{-2x}$.

Case - II

Two real but equal roots. $(m_1 = m_2)$

So, one roots is $y_1 = e^{m_1 x}$. The other indep. roots $y_2 = x e^{m_1 x}$

Check y_2 is also a roots of (1) and $\{y_1, y_2\}$ are independent.

So, the general solⁿ is

$$y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$$

Exm:

$$y'' - 6y' + 9y = 0$$

$$m^2 - 6m + 9 = 0 \implies (m-3)^2 = 0$$

$$m = 3, 3.$$

$$y = C_1 e^{3x} + C_2 x e^{3x}$$

Case-III: Complex roots.

$$m = p \pm iq \qquad \text{if } (b^2 - 4ac < 0)$$

Then, the solⁿ becomes:

$$y = C_1 e^{(p+iq)x} + C_2 e^{(p-iq)x}$$

$$= e^{px}\left(C_1 e^{iqx} + C_2 e^{-iqx}\right)$$

$$= e^{px}\left(\left(C_1 (\cos(qx) + i\sin(qx)\right) + \left(C_2 \cos(qx) - iC_2 \sin(qx)\right)\right)$$

$$= e^{px}\left(A\cos(qx) + B\sin(qx)\right)$$

Exm:

$$y'' + 2y' + 2y = 0$$

$$m = -1 \pm i$$

$$y = e^{-x}(A\cos x + B\sin x).$$