Lagrange's Mean Value theorem :-

Let, $f:[a,b] \to \mathbb{R}$ be a fn.

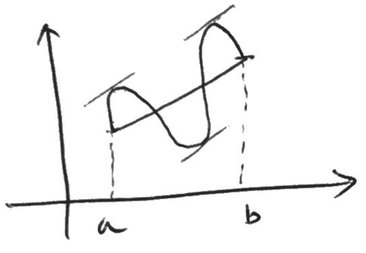
i) f is cont. on $[a,b]$.

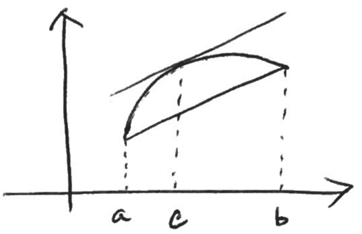
ii) f is derivable in (a,b).

Then, $\exists$ at least one point, $c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

- Geometric :





Proof :-

$$F(x) = f(x) + \lambda x \quad , \quad a \le x \le b$$

$$F(a) = F(b) \Rightarrow f(a) + \lambda a = f(b) + \lambda b$$

$$\Rightarrow \lambda = -\frac{f(b)-f(a)}{b-a}$$

By Rolle's theorem (Check!)

$$F'(c) = 0 \quad , \quad c \in (a,b)$$

$$\Rightarrow f'(c) = -\lambda = \frac{f(b)-f(a)}{b-a}$$

$$\boxed{f(a) = f(b) + \text{~~Ro~~ Lagrange} \equiv \text{Rolle's}}$$

H.W f is cont on $[a,b]$ & diff. in (a,b) with $f'(x) = 0$ in (a,b). Then, f is const. in $[a,b]$.

H.W If $f'(x) = g'(x)$ in $[a,b]$. Then, $f(x) = g(x) + c$. in $[a,b]$

Find a fn. $f(x)$ whose derivative is $\sin(x)$ and whose graph passes through $(0,2)$.

$$g(x) = -\cos x \quad \Rightarrow g'(x) = \sin(x)$$

$$f'(x) = g'(x)$$

$$\Rightarrow f(x) = -\cos x + c$$

$$\Rightarrow 2 = -1 + c$$

$$\Rightarrow c = 3 \quad \Rightarrow \underline{f(x) = 3 - \cos x}$$

Cauchy mean ~~R~~ value thm :-

Let, $f : [a,b] \to \mathbb{R}$ & $g : [a,b] \to \mathbb{R}$ are ~~no~~ two fns.

i) both are continuous on $[a,b]$

ii) both are derivable in (a,b).

iii) $g'(x) \neq 0$ in (a,b).

Then, $\exists$ ~~c~~ at least one $c \in (a,b)$ s.t.

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Proof :-

$$F(x) = f(x) + \lambda g(x).$$

$$F(a) = F(b). \Rightarrow \lambda = -\frac{f(b) - f(a)}{g(b) - g(a)}.$$

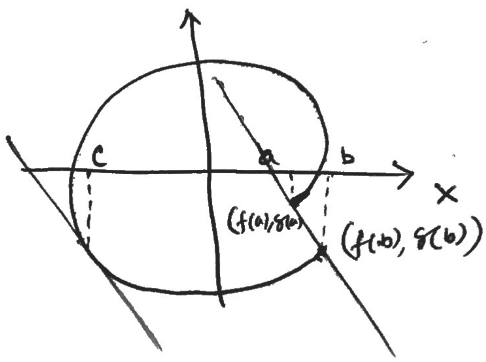
[$g(a) \neq g(b)$, since else $g'(c) = 0$ for some c.]

Then use Rolle's thm.

Geometric :-

$$G : [a,b] \to \mathbb{R}^2$$

$$G(x) = (f(x), g(x))$$



Taylor's theorem with remainder :-

~~$f: I[a,b] \to \mathbb{R}$ be a function. $a \in I$.~~

~~1) $f^{(n)}$ is continuous~~

Let, $f : [a,b] \to \mathbb{R}$ be a function. $x_0 \in [a,b]$.

1) $f^{(n)}$ is continuous on $[a,b]$

11) $f^{(n+1)}$ exists in (a,b).

Then, $\exists$ a point $c \in (x_0, x)$ such that,

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x)$$

where, $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$ is the remainder (Lagrange's form of remainder)

\# If $x_0 = 0$, we get,

$$f(x) = f(0) + f'(0)x + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \frac{x^{n+1}}{(n+1)!}f^{(n+1)}(c),$$

$0 < c < x$

This is called the Maclaurin's theorem.

Exm: $f(x) = e^x$.

$f(0) = 1$, $f'(0) = 1$, ..., $f^{(n)}(0) = 1$

$$f(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}e^c, \quad 0 < c < x$$

Try $f(x) = \sin(x)$