Indeterminate form :-

$\Rightarrow$ $\lim_{x\to a} f(x) = l$, $\lim_{x\to a} g(x) = m \; (\neq 0)$

$$\Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{l}{m}.$$

$\Rightarrow$ If $m=0$ and $l \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ cannot exist.

Because, let, $\lim_{x\to a} \frac{f(x)}{g(x)} = t$. then,

$$l = \lim_{x\to a} f(x) = \lim_{x\to a} \frac{f(x)}{g(x)} \cdot \lim_{x\to a} g(x)$$

$$= t.m$$

$$= 0., \text{ a contradiction.}$$

$\Rightarrow$ Let, $l = m = 0$. Then, $\lim_{x\to a} \frac{f(x)}{g(x)}$ may exist, may not exist or infinite.

Indeterminate forms :- $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0.\infty$, $\infty - \infty$, 0^0, 1^∞, ∞^0

[Not 0^∞, $\infty + \infty$, ∞^∞, $\infty^{-\infty}$]

L' Hospital Rule 1 :- If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ and $f'(a)$, $g'(a)$ exist with $g'(a) \neq 0$., then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Proof :-

$$\frac{f'(a)}{g'(a)} = \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}}$$

$$= \lim_{x\to a} \frac{f(x)}{g(x)}, \text{ since } f(a) = \lim_{x\to a} f(x) = 0$$

$g(a) = 0$ by continuity.

L' Hospital Rule 2 (General form)

~~Let If $f(a) = g(a) = 0$ and f and g are differentiable on an open interval containing a~~

Let, f and g are differentiable on an open interval I Containing a, and $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$. Suppose $g'(x) \neq 0$ for all x in I with $x \neq a$ and

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = l.$$

Then,
$$\lim_{x \to a} \frac{f(x)}{g(x)} = l.$$

Exm :-

$$\lim_{x \to \infty} \left[\sqrt[n]{(x+a_1)(x+a_2)\cdots(x+a_n)} - x \right] \qquad (\infty - \infty)$$

$$= \lim_{x \to \infty} x \left\{ \sqrt[n]{(1+a_1/x)(1+a_2/x)\cdots(1+a_n/x)} - 1 \right] \qquad (\infty \cdot 0)$$

$$= \lim_{x \to \infty} \frac{\sqrt[n]{(1+a_1/x)\cdots(1+a_n/x)} - 1}{1/x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{y \to 0+} \frac{\sqrt[n]{(1+a_1 y)\cdots(1+a_n y)} - 1}{y} \qquad \underline{\frac{1}{x} = y}$$

l'H Rule
$$= \lim_{y \to 0+} \left[\frac{Z}{n} \left\{ \frac{a_1}{1+a_1 y} + \cdots + \frac{a_n}{1+a_n y} \right\} \right] \qquad \left[Z = \sqrt[n]{(1+a_1 y)\cdots(1+a_n y)} \right]$$

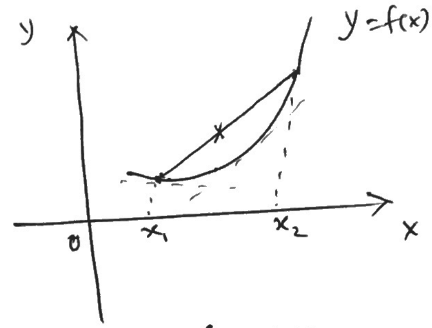
$$= \frac{1}{n} \sum_{i=1}^{n} a_i$$

H.W Show $\lim_{x \to 0+} (1+x)^{1/x} = e$

Concavity & Convexity :

Convex (concave upward)

$f:[a,b] \to \mathbb{R}$ is a function.



Convex

Let, $\frac{df}{dx}$ and $\frac{d^2f}{dx^2}$ are continuous.

Defn f is convex if $f'(x)$ is an increasing function. i.e. $f''(x) \geq 0 \ \forall x$

Or f is convex if the line segment between any two points on the graph lies above the graph.

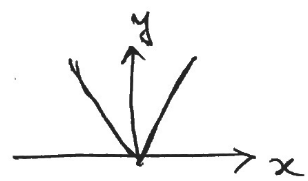
~~Concave~~ Exm :- 1) $f(x) = x^4$

$f''(x) = 12x^2 > 0$; Convex

II) $f(x) = |x|$

f is not derivable at $x = 0$

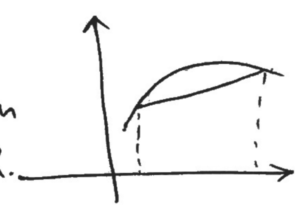
But convex.



H.W. $f(x) = \frac{1}{x}$. (Hint: $(0, \infty)$, $(-\infty, 0)$)

Concave : f is concave if $f'(x)$ is a decreasing function, i.e. $f''(x) \leq 0 \ \forall x$.

Or f is concave if the line segment between any two pts. on the graph lies below the graph.



$\Rightarrow$ f is concave if $-f$ is convex.

Exm: $f(x) = \sqrt{x}$, $f''(x) = -\frac{1}{4x^{3/2}} < 0$

$f(x) = \ln(x)$, $f''(x) = -\frac{1}{x^2} < 0$