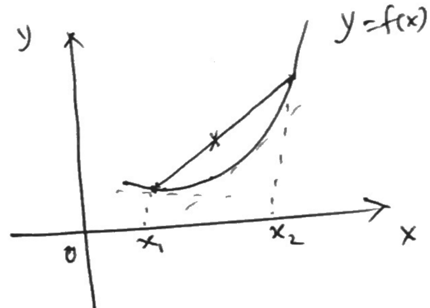
Concavity & Convexity :

Convex (Concave upward)

$f:[a,b] \to \mathbb{R}$ is a function.

Let, $\frac{df}{dx}$ and $\frac{d^2f}{dx^2}$ are continuous.



Convex

Def^n f is convex if $f'(x)$ is an increasing function. i.e. $f''(x) \geq 0 \;\; \forall x$

Or f is convex if the line segment between any two points on the graph lies above the graph.

Exm :- ~~Concave~~

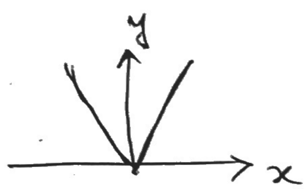
i) $f(x) = x^4$

$f''(x) = 12x^2 > 0$; Convex

ii) $f(x) = |x|$

f is not derivable at $x = 0$

But convex.

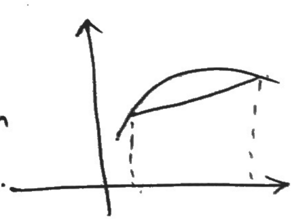


H.W $f(x) = \frac{1}{x}$. (Hint: $(0, \infty)$, $(-\infty, 0)$)

Concave : f is concave if $f'(x)$ is a decreasing function, i.e.

$f''(x) \leq 0. \;\; \forall x.$

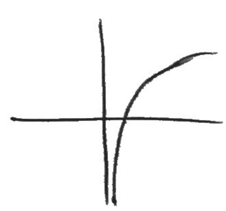
Or f is concave if the line segment between any two pts. on the graph lies below the graph.



$\Rightarrow$ f is concave if $-f$ is convex.

Exm: $f(x) = \sqrt{x}$, $f''(x) = -\frac{1}{4x^{3/2}} < 0$

$f(x) = \ln(x)$, $f''(x) = -\frac{1}{x^2} < 0$



EXM: $f(x) = ax+b$: both Convex & Concave

Point of inflection :-

A point on a curve $y = f(x)$ at which the curve changes from being concave to convex, or vice versa. $(\underline{\underline{f''=0}})$

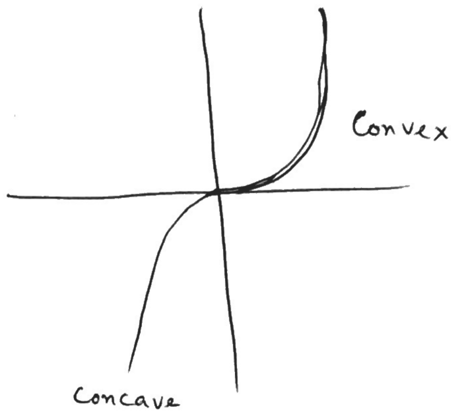
EXM: $y = x^3$

$y' = 3x^2$

$y'' = 6x$

$(-\infty, 0) \rightarrow (0, \infty)$

Concave $\rightarrow$ Convex



$\underline{\underline{x=a}}$ $f''(a-\delta)$ & $f''(a+\delta)$ will be of opposite sign.

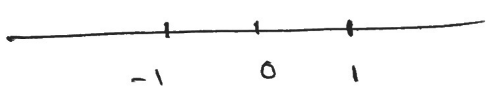
EXM :-

$f(x) = x^4 - 6x^2 + 8x + 10$

$f'(x) = 4x^3 - 12x + 8$

$f''(x) = 12x^2 - 12 = 0$

$\Rightarrow x = \pm 1$



$f''(x) = 12(x+1)(x-1)$

In $(-1-h, -1)$, $f''(x) > 0$, and $(-1, -1+h)$, $f''(x) < 0$

The curve nature is changing from convex to concave at $x = -1$

Similarly, the curve nature is changing from concave to convex at $x = 1$.

So, $x = \pm 1$ ~~are~~ give two point of inflection.

$(1, 13)$ & $(-1, -3)$.

EXM 1.4 10

$f(x) = \cot^{-1}x + x$

$f'(x) = -\frac{1}{x^2+1} + 1$

$f''(x) = \frac{2x}{(1+x^2)^2}$ >0 if $x>0$
<0 if $x<0$

So, concave in $(-\infty,0)$ and convex in $(0,\infty)$.

Therefore, $x=0$ gives a point of inflection, i.e. $(0,\frac{\pi}{2})$

EXM $f(x) = \frac{1}{x}$ $\Rightarrow f'(x) = -1/x^2$

$\Rightarrow f''(x) = -2/x^3$ <0 in $(0,\infty)$
>0 in $(-\infty,0)$

So, $f(x)$ is convex in $(-\infty,0)$ and concave in $(0,\infty)$

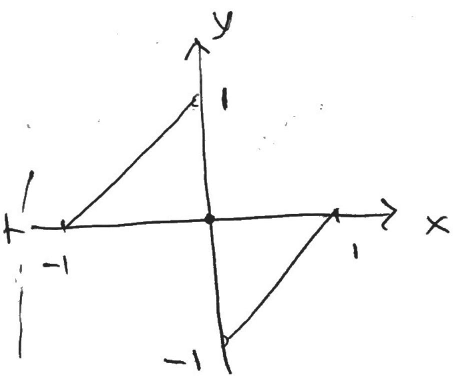
But, $x=0$ is not in the domain. So, there is <u>NO</u> inflection point of f.

Maxima & Minima :-

Extreme value theorem or maxima-minima thm

Continuous fn f is bdd and bounds are attained.
(extreme value)

$$f(x) = \begin{cases} x+1, & -1 \le x < 0 \\ 0, & x=0 \\ x-1, & 0 < x \le 1. \end{cases}$$



The function has ~~heigh~~ neither a highest nor a lowest point.

Absolute maximum (global) / minimum

$f : D \to \mathbb{R}$ is a fn.
$[a,b]$

I) f has an absolute maximum value at c if

$$f(x) \le f(c) \quad \forall x \in D.$$

II) f has an absolute minimum value at d if

$$f(x) \geq f(d) \quad \forall x \in D.$$

Local Extremum :-

I) f is said to have a local maximum at $c \in (a,b)$

if $\exists\ \delta > 0$ such that

$$f(x) \le f(c) \quad \forall x \in (c-\delta, c+\delta) \cap [a,b]$$

II) f is said to have a local minimum at $c \in (a,b)$

if $\exists\ \delta > 0$ such that,

$$f(x) \geq f(d) \quad \forall x \in (c-\delta, c+\delta) \cap [a,b].$$

$$* \begin{cases} \text{Absolute max} = \max\{f(a), f(b), \text{local } \text{max}\} \\ \text{Absolute min} = \min\{f(a), f(b), \text{local min}\} \end{cases}$$

First derivative test :-

\# If f has a local maximum or minimum (extremum) at an interior point c, and if $f'(c)$ exists, then

$$f'(c) = 0$$

Proof :- f has local max $\Rightarrow$ $\exists\ \delta > 0$ s.t.

$$f(x) \le f(c) \quad \forall x \in (c-\delta, c+\delta) \cap [a,b]$$

$$\Rightarrow f(x) - f(c) \le 0.$$

Now, $\dfrac{f(x) - f(c)}{x - c} \geq 0$ for $x \in (c-\delta, c) \cap [a,b]$

and $\dfrac{f(x) - f(c)}{x - c} \geq 0$ for $x \in (c, c+\delta) \cap [a,b]$

Now, as $f'(c)$ exists,

$$\lim_{x \to c+} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c-} \frac{f(x) - f(c)}{x - c} = f'(c)$$

This can be only true if

$\underline{f'(c) = 0}$. (Critical points) $\underline{\underline{f'(x) = 0}}$

Max or Min Value

Check:

i) Interior point where $f' = 0$

ii) Interior point where f' is undefined

iii) end pts. $f(a)$, $f(b)$

*

The Theorem/result is necessary, not sufficient!

Exm: 1. $f(x) = x^3$

$\Rightarrow f'(x) = 3x^2 = 0 \Rightarrow x = 0$ (Critical points)

But, $f(x) - f(0) < 0$ in $(-\delta, 0)$

and $f(x) - f(0) > 0$ in $(0, \delta)$

So, f has no local extremum at $x = 0$.

2. $f(x) = |x|$

f is not derivable at $x = 0$.

but, $f(x) \geq f(0) = 0 \;\forall x$.

So, f has a local (global) extremum at $x = 0$.

~~EXAM~~

2nd derivative test:-

Let, f is diff. at c and $f'(c)=0$.

Let, $f''(x)$ exists and Continuous in a nbd of c.

Then,

1) f has a local maximum at $x=c$, if $f''(c)<0$.

11) f has a local minimum at $x=c$, if $f''(c)>0$.

If $f''(c)=0$, then, no conclusive decision on that stage.

General form:-

Let, $f:[a,b]\to\mathbb{R}$ be a function and $c\in(a,b)$ such that,

$$f'(c)=f''(c)=\cdots=f^{(n-1)}(c)=0 \quad \text{and} \quad f^{(n)}(c)\neq 0.$$

Then, f has

1) a local maximum when $n=$ even & $f^{(n)}(c)<0$

11) a local minimum when $n=$ even & $f^{(n)}(c)>0$

11) no extremum if $n=$ odd.

FT

21 $f(t)=2-|t|$, $-1\le t\le 3$

$$f(t)=\begin{cases} 2+t, & -1\le t<0 \\ 2, & t=0 \\ 2-t, & 0<t\le 3 \end{cases}$$

At $t=0$, $f(t)$ is not differentiable.

In $(-1,0)$, $f'(t)=1$, so no extremum

In $(0,3)$, $f'(t)=-1$, so no extremum.

Cont. at 0 & end pt. Therefore absolute extremum $=\max/\min\{f(-1), f(0), f(3)\}$

$=\max/\min\{1, 2, -1\}$

So, absolute maximum at $x=0$ and absolute minimum at $x=3$.