§ Calculus of Several Variables

$y = f(x)$. $\quad f: D \to \mathbb{R}$ or $f: \mathbb{R} \to \mathbb{R}$.

$z = f(x, y)$: $f: D_1 \times D_2 \to \mathbb{R}$ or $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$f(x, y)$

or

$f: \mathbb{R}^2 \to \mathbb{R}$.

In general,

$f: \mathbb{R}^n \to \mathbb{R}$: $z = f(x_1, x_2, \dots, x_n)$: explicit fn.

n-tuples.

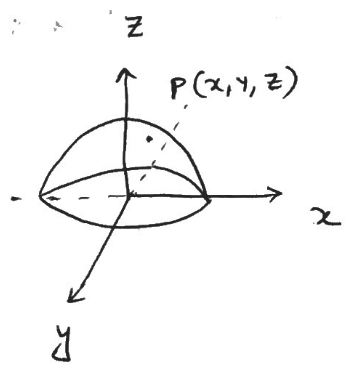
$g(z, x_1, x_2, \dots, x_n) = 0$: Implicit function

x_i : independent variables

z : dependent variable.

<u>Exm</u>:

$z = \sqrt{x^2 + y^2}$



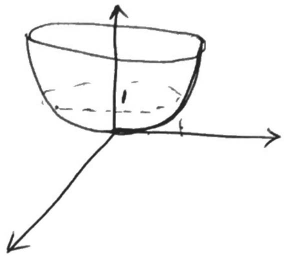
	Domain	Range
$z = \sqrt{x - y^2}$	$x \geqslant y^2$	$[0, \infty)$
$z = \frac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$z = \cos(xy)$	$\mathbb{R}^2$	$[-1, 1]$.
$z = \log(x + y)$	$x + y > 0$	$\mathbb{R}$

<u>Distance</u>: $P(x_0, y_0)$ $\quad Q(x_1, y_1)$

$$d(P, Q) = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}.$$

$Z = x^2+y^2 = c$, Constant. Level curve.

$Z = f(x,y)$: Surface



Neighbour hood of a point :-

The δ- nbd of a point $P(x_0, y_0)$ is the set of all points $Q(x,y)$ such that,

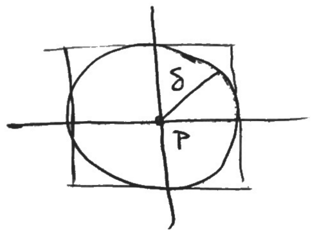
$$d(P,Q) < \delta.$$

i.e. $$\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$$

$$N_\delta(P) = \left\{ (x,y) \in \mathbb{R}^2 \mid \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta \right\}$$

Or $|x-x_0| < \delta$ and

$|y-y_0| < \delta$



Limit point :-

A point $P(x_0,y_0) \in \mathbb{R}^2$ is said to be a limit point of $D \subseteq \mathbb{R}^2$ if $\exists$ a $\delta > 0$ such that, $N_\delta(P)$ has at least one point of D.

Let, $f: D \to \mathbb{R}$ and $P(x_0,y_0)$ be a limit pt. of D. $D \subset \mathbb{R}^2$

Limit of a function :-

We say $f(x,y)$ approaches to L as (x,y) approaches to (x_0,y_0) if for any fix $\epsilon > 0$, $\exists\, \delta > 0$ such that,

$$|f(x,y) - L| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta.$$

We denote by $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$.

$\Rightarrow \lim_{(x,y)\to(x_0,y_0)} f(x,y)$, if exists, is unique!

Product Rule :-

$\lim_{\substack{(x,y)\to(x_0,y_0)\\ z \quad z_0}} f(x,y) = L \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$

P. T. $\lim_{(x,y)\to(x_0,y_0)} f(x)\, g(x) = LM$.

$\lim_{z\to z_0} \left(f(x,y) - L\right) = 0 \qquad \lim_{z\to z_0} \left(g(x,y) - M\right) = 0.$

So, $|f(x,y) - L| < \sqrt{\epsilon}$ whenever $0 < d(z,z_0) < \delta_1$

$|g(x,y) - M| < \sqrt{\epsilon}$ whenever $0 < d(z,z_0) < \delta_2$

Choose, $\delta = \min\{\delta_1, \delta_2\}$. If $0 < d(z,z_0) < \delta$, then,

$$\left|\left(f(x,y)-L\right)\left(g(x,y)-M\right) - 0\right| = |f(x,y)-L|\,|g(x,y)-M| < \epsilon.$$

So, $\lim_{z\to z_0} \left(f(x,y)-L\right)\left(g(x,y)-M\right) = 0$.

Now, $\left(f(x,y)-L\right)\left(g(x,y)-M\right) = fg - Lg - Mf + LM$

$\Rightarrow fg = (f-L)(g-M) + Lg + Mf - LM$

So, $\lim_{z\to z_0} f(x,y)g(x,y) = \lim_{z\to z_0} (f-L)(g-M) + \lim_{z\to z_0} L\, g(x,y) + \lim_{z\to z_0} M f(x,y) - LM$

$= 0 + LM + LM - LM = LM.$

EXM :-

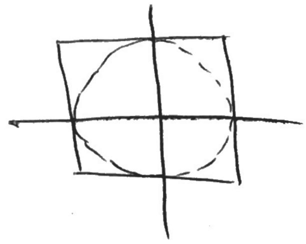
To prove $\lim_{(x,y)\to(0,0)} \left[y + x\cos\left(\frac{1}{y}\right)\right] = 0$

Not defined along $y = 0$. But $(0,0)$ is a limit pt.

$$\left|y + x\cos\left(\tfrac{1}{y}\right)\right| \le |y| + |x|\left|\cos\left(\tfrac{1}{y}\right)\right| \le |y| + |x| < \epsilon$$

whenever, $0 < |x| < \epsilon/2$ and $0 < |y| < \epsilon/2$

So, $\lim_{(x,y)\to(0,0)} \left[y + x\cos\left(\frac{1}{y}\right)\right] = 0$



$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\sin\left(\frac{1}{xy}\right) = 0.$$

$$\left|(x^2+y^2)\sin\left(\tfrac{1}{xy}\right)\right| \le (x^2+y^2) < \epsilon$$

whenever $0 < \sqrt{x^2+y^2} < \delta = \sqrt{\epsilon}$.

Polar Rule :-

Substitute $x = r\cos\theta$, $y = r\sin\theta$ and see the limit as $r \to 0$

$$\frac{x^2}{x^2+y^2} = \frac{r^2\cos^2\theta}{r^2} = \cos^2\theta$$ takes on values from 0 to 1 regardless of $r \to 0$. So $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ doesn't exist.

Two path test :-

If $f(x,y)$ has different limits along two different paths as $(x,y) \to (x_0, y_0)$, then $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ doesn't exist.

EXM. $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy}$ Consider the path $y = mx$, $x \neq 0$.

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=mx}} f(x,y) = \lim_{(x,y)\to(0,0)} f(x,y)\Big|_{y=mx} = \frac{1+m^2}{m}$$

So, the limit is different for $y=x$ and $y=-x$.

Hence, the limit does not exist.

Continuity :-

A function $f(x,y)$ is continuous at the point (x_0, y_0) if

i) $f(x_0, y_0)$ is defined

ii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$

Exm:-

$$f(x,y) = \begin{cases} x\tan^{-1}\left(\frac{y}{x}\right), & (x,y) \neq (0,1) \\ 0, & (x,y) = (0,1). \end{cases}$$

$$\lim_{\substack{x\to 0+\\ y\to 1}} x\tan^{-1}\left(\frac{y}{x}\right) = 0 = f(0,1)$$

and $$\lim_{\substack{x\to 0-\\ y\to 1}} x\tan^{-1}\left(\frac{y}{x}\right) = 0 = f(0,1).$$

Alltogether, $$\lim_{(x,y)\to(0,1)} x\tan^{-1}\left(\frac{y}{x}\right) = 0 = f(0,1)$$

So, f is continuous at $(0,1)$.

Exm:

$$f(x,y) = \begin{cases} \frac{x(x^2-y^2)}{(x^2+y^2)}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$|f(x,y) - f(0,0)| \leq \frac{|x|(x^2+y^2)}{(x^2+y^2)} = |x| < \epsilon$$

whenever, $|x| \leq \sqrt{x^2+y^2} < \delta = \epsilon$.

So, $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$

So, f is continuous at $(0,0)$.

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$$f(x,y) = \begin{cases} \dfrac{x^2-2xy+y^2}{x-y}, & (x,y) \neq (1,-1) \\ 0, & (x,y) = (1,-1) \end{cases}$$

$$\lim_{(x,y)\to(1,-1)} f(x,y) = \lim_{(x,y)\to(1,-1)} \frac{(x-y)^2}{(x-y)} = \lim_{(x,y)\to(1,-1)} (x-y)$$

$$= 2 \neq f(1,-1)$$

So, not continuous at $(1,-1)$.

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$$f(x,y) = \begin{cases} \dfrac{x^4y^4}{(x^2+y^4)^3}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Let, $(x,y) \to (0,0)$ along $x = my^2$, $y \neq 0$.

then,

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } x=my^2}} f(x,y) = \lim_{y\to 0} f(x,y)\Big|_{x=my^2} = \frac{m^4}{(m^2+1)^3}$$

which is diff. for diff values of m.

So, the limit does not exist, hence, not continuous

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$$f(x,y) = \begin{cases} \dfrac{2x^2+y^2}{3+\sin x}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$|f(x,y) - f(0,0)| = \left|\frac{2x^2+y^2}{3+\sin x}\right| \leq \frac{1}{2}|2x^2+y^2| \leq x^2+y^2 < \epsilon$$

whenever $\sqrt{(x^2+y^2)} < \sqrt{\epsilon} = \delta$.

So, $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$.

Hence, f is continuous at $(0,0)$

27.

$$f(x,y) = \begin{cases} \dfrac{xy(x-y)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$|f(x,y) - f(0,0)| = \frac{|xy|\,|x-y|}{|x^2+y^2|} \leq \frac{1}{2}(|x|+|y|) < \epsilon$$

whenever $|x| < \delta$ and $|y| < \delta$ with $\delta = \epsilon$.