Complex Analysis (MATH-2)

BCM Math-2

Books :-

1. Advanced Engineering Mathematics, E. Kreyszig
2. " , Jain & Iyenger
3. Complex Variables & Applications, Brown, Churchill.

Class Notes : bankimmath.weebly.com/lecnotes.html

Class Tests: Section II : 2nd April.
Section III : 5th April.

Complex Numbers :-

$\mathbb{N}$: natural numbers. $\quad \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

$\downarrow$

$x+2=1 \;:\; \mathbb{Z}$: integers.

$\downarrow$

$2x-1=2 \;:\; \mathbb{Q}$: rational numbers.

$\downarrow$

Quadratic eqns.: $x^2-2=0$, $\quad \mathbb{R} = \mathbb{Q} \cup \underbrace{(\mathbb{R}-\mathbb{Q})}_{\mathbb{I}}$

Real numbers.

$\downarrow$

Limits $\quad \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$: Extended Real numbers.

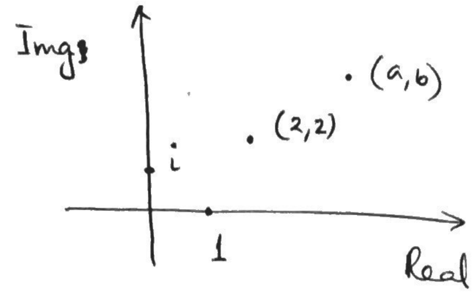
$\downarrow$

$x^2+1=0$ has no real solutions.

We must 'invent' a number whose square is -1; denote it $i \; (=\sqrt{-1})$ and call it imaginary unit. (Sometime j)

Also, $(-i)^2 = -1$. So, $x^2+1=0$ has two roots $\pm i$.

$\Rightarrow$ Imaginary numbers have square a neg. value : $-2i$



A complex number z is an ordered pair (x, y) of reals x, y. i.e.

$$z = (x, y)$$

- x is called the real part of z, $x = \mathrm{Re}\, z$
- y is called the imaginary part of z, $y = \mathrm{Im}\, z$.

$$1 = (1,0) \quad , \quad i = (0,1).$$

Now, $(x,y) \equiv x.(1,0) + y(0,1)$

$$= x.1 + y.i = \underline{x + iy}$$

<u>Addition & Multiplication of two complex Numbers :</u>

$$z_i = (x_i, y_i)$$

1) $z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$

II) $z_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$

$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$

$(x_1, 0) \cdot (x_2, 0) = (x_1 x_2, 0)$

So, the set of complex numbers $\mathbb{C}$ extends the real numbers.

$\Rightarrow \quad i = (0,1)\cdot(0,1) = (-1,0) = -1.$

Therefore in $\mathbb{C}$: $\quad z^2+1 = (z+i)(z-i)$

In general: $\quad z^2+\omega^2 = (z+i\omega)(z-i\omega).$

Reciprocal Formula :-

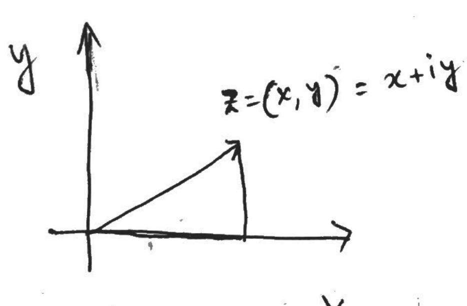
$$\frac{1}{a+ib} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - i\,\frac{b}{a^2+b^2}.$$

$$\equiv \left(\frac{a}{a^2+b^2},\ -\frac{b}{a^2+b^2}\right).$$

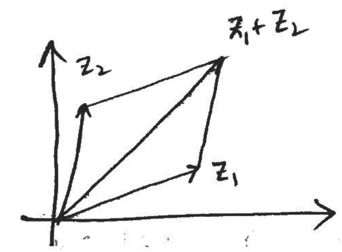
Defn :- If $z = x+iy$, then, $|z| = \sqrt{x^2+y^2}$ is called the **absolute value** of z and $\bar{z} = x-iy$ is called the **complex conjugate** of z.

Also, $z.\bar{z} = (x^2+y^2) = |z|^2.$

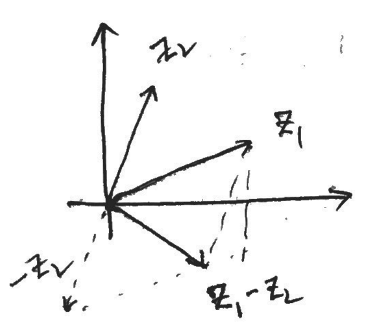
Argand diagram or Complex plane :-



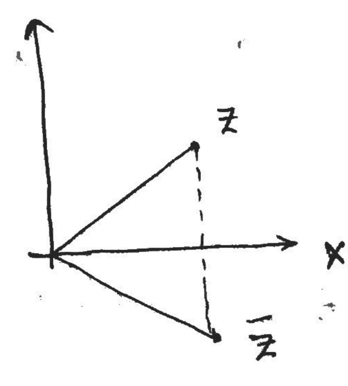
z



z_1+z_2 (Parallelogram law)



z_1-z_2



Reflection across x-axis.

$\Rightarrow \quad z = x+iy, \quad x, y \in \mathbb{R}$

$$|z| = \sqrt{x^2+y^2} \geqslant x = \operatorname{Re} z$$
$$\geqslant y = \operatorname{Im} z$$

So, $\quad -|z| \leq \operatorname{Re} z \leq |z|$

$-|z| \leq \operatorname{Im} z \leq |z|$.

~~The~~ Triangle Inequality:

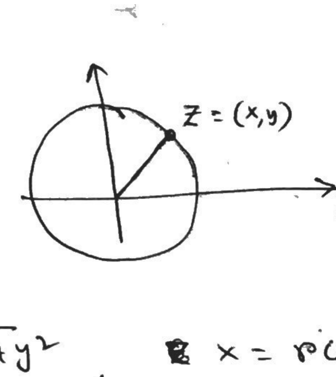
$|z_1 + z_2| \leq |z_1| + |z_2|$, equality holds when $0, z_1$ & z_2 are co-llinear.

Properties :-

$$x = \operatorname{Re} z = \frac{z+\bar{z}}{2}$$

$$y = \operatorname{Im} z = \frac{z-\bar{z}}{2i}$$

Now, Cartesian $\rightarrow$ ~~pla~~ polar co-ordinates.



Let, $r = |z| = +\sqrt{x^2+y^2}$, $x = r\cos\theta$, $y = r\sin\theta$ [r is +ve, r is the distance from the origin]

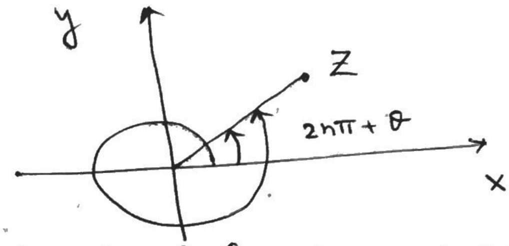
So, $z = x+iy = r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$ (By Euler Identity)

$\Rightarrow \theta$ is called the argument of z, denoted by $\arg z$

$z = 0$: θ is undefined.

For $z \neq 0$, there are infinitely many argument values.



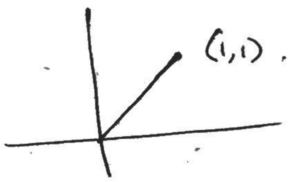
However, one can uniquely determine $\arg z$ for $z \neq 0$ if $-\pi < \theta \leq \pi$. The unique value is called the
$(0 \leq \theta < 2\pi)$
principal argument, $\operatorname{Arg} z$.

Exponential form or ~~pla~~ polar form of complex :-

$$z = re^{i\theta}$$
$$= |z| e^{i \arg z}$$

Exm :- $z = 1 + i$

$|z| = \sqrt{2}$,



$$x = |z| \cos\theta$$
$$y = |z| \sin\theta$$
$$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}}, \quad \sin\theta = \frac{1}{\sqrt{2}}.$$
$$\therefore \theta = \pi/4 \pm 2n\pi. \quad (n = 0, 1 \ldots$$

$$\operatorname{Arg} z = \frac{\pi}{4}.$$

H.W. $z = -1 - i$, $|z| = \sqrt{2}$

$$\operatorname{Arg} z = -\frac{3\pi}{4}.$$

Result :- $x + iy = re^{i\theta}$ lies on the circle of radius $r = |z|$.

If $r = 1$, $e^{i\theta}$ lies on the circle of radius 1. $\forall \theta$.

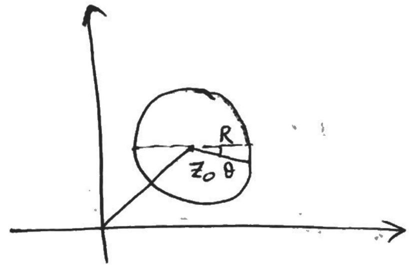
$$e^{i\pi} = -1, \quad e^{-i\pi/2} = -i, \quad e^{-i4\pi} = 1$$

$z = Re^{i\theta}$, $0 \le \theta \le 2\pi$ is a parametric representation of the circle, centered at $(0,0)$, radius R.

By shifting the origin,

$$z = z_0 + Re^{i\theta}, \quad 0 \le \theta \le 2\pi$$

represents a circle, centered at z_0, radius R.



<u>H.W.</u> Calculate $(\sqrt{3}+i)^7 = (2e^{i\pi/6})^7 = 2^7 e^{i7\pi/6}$

$$= (2^6 e^{i\pi})(2e^{i\pi/6})$$

$$= -64(\sqrt{3}+i)$$

$\Rightarrow$ Easy to multiply, take powers, finding roots of complex numbers in polar forms.

<u>Notation:</u>

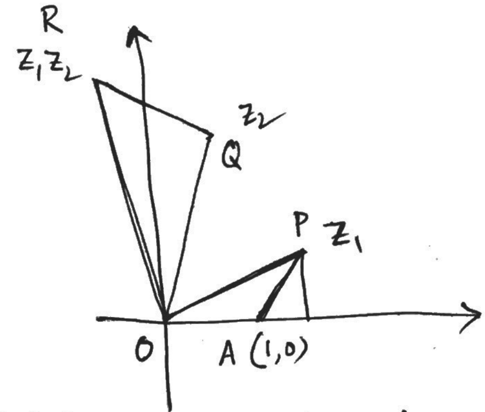
$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

$$= \underline{r\,\mathrm{Cis}\,\theta}$$

Let, $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, $z_1, z_2 \ne 0$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

So, $|z_1 z_2| = r_1 r_2$, $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$



ΔOAP & ΔOQR are similar triangle.

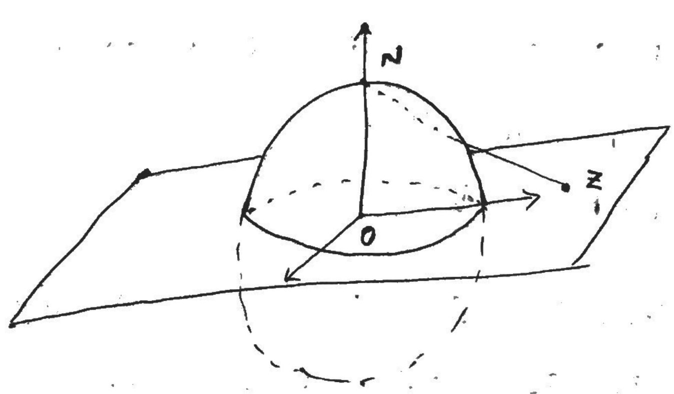
$$\frac{OA}{OQ} = \frac{OP}{OR} = \frac{AP}{QR}.$$

Spherical Representation

Consider a fn. $f : S \to \mathbb{C} \cup \{\infty\}$ by

$$f(x_1, x_2, x_3) = \begin{cases} \dfrac{x_1 + i x_2}{1 - x_3}, & x_3 \neq 1 \\ \infty, & x_3 = 1. \end{cases}$$

$$S = \{(x_1, x_2, x_3) : (x_1^2 + x_2^2 + x_3^2) = 1\}$$



This correspondence is called the Stereographic projection.
The sphere is Riemann sphere.