Calculus

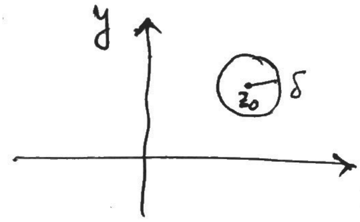
Circle : $z = z_0 + Re^{i\theta}$

$$\Rightarrow |z - z_0| = R|e^{i\theta}| = R$$

i.e. $(x-x_0)^2 + (y-y_0)^2 = R^2$

Neighbourhood of a point :- A δ-neighbourhood of a point $z_0 \in \mathbb{C}$ is the set of all points whose distance from z_0 is less than δ.

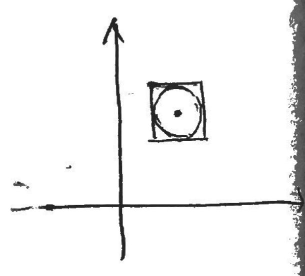
i.e. $|z - z_0| < \delta$



i.e. $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

Or, one can consider a square nbd.

i.e. $|x - x_0| < \delta$, $|y - y_0| < \delta$.



$$N_\delta(z_0) = \{ z \in \mathbb{C} \mid |z - z_0| < \delta \}$$

Bounded Set :- Let, $D \subseteq \mathbb{C}$. D is said to be bounded if $\exists$ $M > 0$ such that

$|z| < M$ $\forall$ $z \in D$.

Limit point :- Let, $z_0 \in D \subseteq \mathbb{C}$. z_0 is said to be a limit point of D if every neighbourhood of z_0, $N_\delta(z_0)$, contains at least one point of D, other than z_0.

Function of Complex Variable :-

Let, $f : D \to \mathbb{C}$ is a function, $D \subseteq \mathbb{C}$.

$$w = f(z).$$

Exm : 1. $f(z) = \frac{1}{z}$, $z \neq 0$

2. $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$: polynomial fn.

3. $f(z) = |z|^2$

z and $f(z)$ are two complex numbers.

Let, $z = x + iy \equiv (x, y)$

& $f(z) = \cancel{\ } u + iv$.

Then, $u + iv = f(x + iy)$.

We can think of $f : \mathbb{R}^2 \to \mathbb{R}^2$.

Exm: $f(z) = |z|^2$, $g(z) = z^2$

$f(x+iy) = x^2 + y^2$, $g(z) = x^2 - y^2 + 2ixy$

$= u(x,y) + iv(x,y)$ | $= u(x,y) + i\,v(x,y)$

$\Rightarrow v = 0$, $u = x^2 + y^2$ | $u(x,y) = x^2 - y^2$, $v(x,y) = 2xy$.

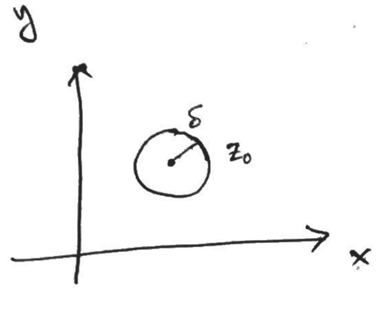
If $v(x,y) = 0$, we can think f as real-valued function. $f : \mathbb{R}^2 \to \mathbb{R}$.

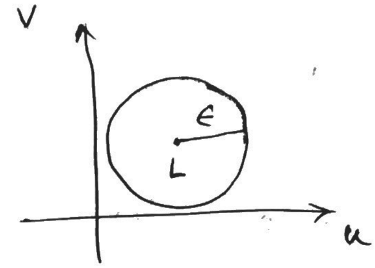
H.W $f(z) = 2z + \bar{z}$. Find u & v at $z_0 = 1 + i$

$\Rightarrow$ $u = Re\, f(z)$, $v = Re\, f(z)$.

<u>Limit</u> :- Let, $f: D \to \mathbb{C}$ be a function and z_0 is a limit point of D. Then, f is said to have a limit $L \in \mathbb{C}$ as $z \to z_0$ if for any $\epsilon > 0$, $\exists\, \delta > 0$ such that

$$|f(z) - L| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta, \; z \in D.$$





\# We denote it as

$$\lim_{z \to z_0} f(z) = L.$$

\# If $\lim_{z \to z_0} f(z)$ exists, it is unique.

<u>Hint</u> : Suppose not, $L_1 = \lim_{z \to z_0} f(z)$ and $L_2 = \lim_{z \to z_0} f(z)$.

Prove $|L_1 - L_2| < \epsilon \quad \forall\, \epsilon > 0$

<u>Exm</u> : $\lim_{z \to 1} i\dfrac{\bar{z}}{2} = \dfrac{i}{2}$ $\Big\|$ $\lim_{(x,y) \to (1,0)} \dfrac{y + ix}{2} = \dfrac{i}{2}$.

$D_1 = \{|z| < 1\}$ ✓

$D_2 = \{|z| < \frac{1}{2}\}$, ✗

$$\left|f(z) - \frac{i}{2}\right| = \left|\frac{i\bar{z}}{2} - \frac{i}{2}\right| = \frac{|\bar{z} - 1|}{2} = \frac{\sqrt{(x-1)^2 + y^2}}{2} < \epsilon$$

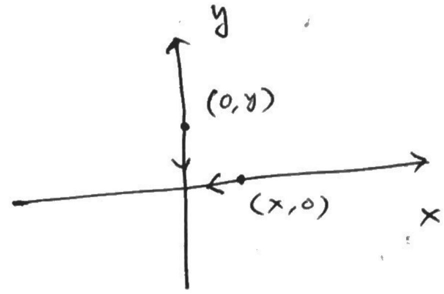
whenever, $\sqrt{(x-1)^2 + y^2} = |z - 1| < \delta \; (= 2\epsilon)$.

So, $\lim_{z \to 1} i\dfrac{\bar{z}}{2} = \dfrac{i}{2}$.

Exm. $f(z) = \dfrac{z}{\bar{z}}$, $z \neq 0$. Does $\lim_{z\to 0} f(z)$ exist?

$$\frac{z}{\bar{z}} = \frac{x+iy}{x-iy} = \frac{x^2-y^2+2ixy}{x^2+y^2}$$

$$\lim_{z\to 0} f(z) = \lim_{(x,y)\to(0,0)} \frac{x^2-y^2+2ixy}{x^2+y^2}$$



Choose $z = (x,0)$ that approches to $(0,0)$.

$$\lim_{\substack{(x,0)\to(0,0) \\ \text{along } x\text{-axis}}} \frac{z}{\bar{z}} = 1.$$

Choose $z = (0,y)$ that approches to $(0,0)$.

$$\lim_{\substack{(0,y)\to(0,0) \\ \text{along } y\text{-axis}}} \frac{z}{\bar{z}} = -1.$$

Since the limit is unique, we must conclude that the limit does not exist.

Result :- $z = x+iy$, $f(z) = u(x,y) + iv(x,y)$

Let, $L = L_1 + iL_2$, $z_0 = (x_0, y_0)$.

$\lim_{z\to z_0} f(z) = L$ iff $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = L_1$ and

$$\lim_{(x,y)\to(x_0,y_0)} v(x,y) = L_2.$$

Continuity :- Let, $f : D \to \mathbb{C}$ and $z_0 \in D$. f is said to be continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

limit at infinity : $f(z)$ has limit L as $z \to \infty$ if for every $\epsilon > 0$, $\exists$ $M > 0$ such that

$$|f(z) - L| < \epsilon \quad \text{whenever } |z| > M.$$

Limit to infinity : $f(z)$ going to infinity as $z \to z_0$ if for every $M > 0$, $\exists$ $\delta > 0$ such that,

$$|f(z)| > M \quad \text{whenever } 0 < |z - z_0| < \delta.$$

Result :- $\lim_{z \to z_0} f(z) = \infty$ iff $\lim_{z \to z_0} \frac{1}{f(z)} = 0$

$\lim_{z \to \infty} f(z) = L$ iff $\lim_{z \to 0} f\left(\frac{1}{z}\right) = L$.

f is said to continuous on D if f is continuous at each $z \in D$.

Exm: ~~lim~~ $f(z) = x + iy^2$ is continuous at $z = 2i$.

$f(2i) = 4i$

Now, $|f(z) - f(2i)| = |x + iy^2 - 4i|$

$$\leq |x| + |y+2||y-2|$$

$$\leq |x| + (4+\delta)(|y-2|) < \delta + (4+\delta)\delta < \epsilon$$

with $|x| < \delta$, $|y-2| < \delta$, whenever, $\delta^2 + 5\delta < \epsilon$.

i.e. Whenever $\delta < \sqrt{\epsilon + \frac{25}{4}} - \frac{5}{2}$.

With this δ, the result is true. So, f is cont. at $z = 2i$.

Exm. $\lim_{z \to \infty} \frac{z}{3-z}$

$$= \lim_{z \to 0} \frac{1/z}{3 - 1/z} = \lim_{z \to 0} \frac{1}{3z - 1} = -1.$$

H.W. $\lim_{z \to 0} \frac{z}{|z|}$. Choose path along $x = 0$ & $y = 0$.

Result: Let, $f(z) = u(x,y) + i v(x,y)$. f is cont. at $z_0 = (x_0, y_0)$ iff u & v are cont. at (x_0, y_0).

Exm: $e^z = e^{x+iy} = e^x \cos y + i e^x \sin y = u + iv$.

u, v are cont. at (x,y), so e^z is continuous.

H.W. $f(z) = \begin{cases} \frac{\text{Im}(z^2)}{|z|^2}, & z \neq 0 \\ 0, & z = 0 \end{cases} = \begin{cases} \frac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Path along $y = mx$. $f(z) = \frac{2m}{1+m^2}$, diff. for diff m.

Thus, Limit does not exist.

H.W $\lim_{z \to 0} \left(\frac{z}{\bar{z}}\right)^2$. Path along ~~[illegible]~~ $y = 0$ and along $y = x$.

~~Result~~

Derivative :- Let, $f : D \to \mathbb{C}$, $D \subseteq \mathbb{C}$. The derivative of f at $z_0 \in D$ is denoted by $f'(z_0)$ and is ~~denote~~ defined by the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \text{ provided it}$$

exists. Then, f is said to be differentiable at z_0.

$$\text{Or, } \quad f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}.$$

Result :- If f is differentiable at z_0, f is continuous at z_0.

Hint: $$f(z) - f(z_0) = \frac{f(z) - f(z_0)}{(z - z_0)} \cdot (z - z_0)$$

$$\therefore \lim_{z \to z_0} f(z) = \lim_{z \to z_0} \left\{ \frac{f(z) - f(z_0)}{z - z_0} (z - z_0) + f(z_0) \right\}$$

$$= f'(z_0) \cdot \lim_{z \to z_0} (z - z_0) + f(z_0)$$

$$= f(z_0)$$

Counter Example $$f(z) = \bar{z} = x - iy, \quad z_0 = (0,0).$$

$$= u(x,y) + i\,v(x,y)$$

u, v are cont. $\Rightarrow f$ is cont.

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{(x,y) \to (0,0)} \frac{x - iy}{x + iy} = \lim_{z \to 0} \frac{\bar{z}}{z} \text{ does not exist.}$$

Exm $f(z) = |z|^2 = x^2+y^2$
$= u(x,y) + i v(x,y)$.

$$\left.\begin{aligned} u(x,y) &= x^2+y^2 \\ v(x,y) &= 0 \end{aligned}\right\} \text{ cont. at } (0,0) \text{ or any } (x,y).$$

So, $f(z)$ is cont. at $(0,0)$ and at any z.

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{z\to 0} \frac{|z|^2}{z} = \lim_{z\to 0} \frac{z\bar{z}}{z} = 0$$

So, diff. at $z_0 = 0$ and $f'(0) = 0$.

Let, $z_0 \neq 0$:

$$\cancel{\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{z\to z_0} \frac{|z|^2-|z_0|^2}{z-z_0}}$$

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = \lim_{h\to 0} \frac{(z_0+h)(\bar{z}_0+\bar{h}) - z_0\bar{z}_0}{h}$$

$$= \lim_{h\to 0} \frac{h\bar{z}_0 + \bar{h} z_0 + h\bar{h}}{h}$$

$$= \lim_{h\to 0} \left(\overline{z_0+h} + z_0 \frac{\bar{h}}{h} \right) \text{ does not exist.}$$

So, $f(z) = |z|^2$ is not diff. at $z \neq 0$.

Analytic function :- A fn $f(z)$ is said to be analytic in D if ~~$f(z)$~~ f is defined and differentiable at all points of D.

$\Rightarrow$ f is said to be analytic at a point $z_0 \in D$ if f is analytic in a neighbourhood of z_0.

i.e. $\exists\ \delta > 0$ s.t f is analytic in $N_\delta(z_0)$.