EXM : $f(z) = z^n$ is analytic in $\mathbb{C}$.

Counter Example :-

$f(z) = |z|^2$ is not analytic at $z = 0$.

Cauchy - Riemann Equations :-

Thm Suppose $f(z) = u(x,y) + i\,v(x,y)$ is cont. in a nbd of z_0 and f' exists at $z_0 = (x_0, y_0)$. Then, the first-order P.ds. of u and v exist at (x_0, y_0) and they satisfy:

(1) $u_x = v_y$, $u_y = -v_x$ at (x_0, y_0);

and $f'(z_0) = u_x + i v_x = v_y - i u_y$.

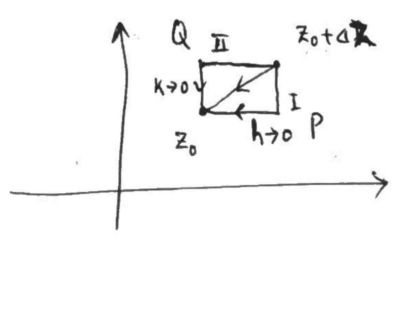
If f is analytic in D, then these P.ds exist and satisfy (1) at all pts. of D.

$\Rightarrow$ The eqn (1) is called the Cauchy - Riemann eqn.

Hint : f is diff. at $z_0 = (x_0, y_0)$. So,

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{(h,k)\to(0,0)} \frac{u(x_0+h, y_0+k) + i\,v(x_0+h, y_0+k) - u(x_0,y_0) - i\,v(x_0,y_0)}{h+ik}$$



Since the limit exists, it should be indep. of path we choose.

Path I : $z_0 + \Delta z \to z_0$ or $(h,k) \to (0,0)$ through $(h,0)$.
[First $k \to 0$, then $h \to 0$]

then,

$$f'(z_0) = \lim_{h\to 0} \frac{u(x_0+h, y_0) - u(x_0,y_0)}{h} + \lim_{h\to 0} i \frac{v(x_0+h,y_0) - v(x_0,y_0)}{h}$$

$$= u_x(x_0,y_0) + i\, v_x(x_0,y_0).$$

Path II $(h,k) \to (0,0)$ through $(0,k)$. [First $h \to 0$, then $k \to 0$]

$$f'(z_0) = \lim_{k\to 0} \frac{u(x_0, y_0+k) - u(x_0,y_0)}{ik} + \lim_{k\to 0} \frac{v(x_0,y_0+k) - v_0(x_0,y_0)}{k}$$

$$= -i\, u_y(x_0,y_0) + v_y(x_0,y_0)$$

$$= v_y(x_0,y_0) - i\, u_y(x_0,y_0)$$

The limit should be unique! So,

$$u_x = v_y, \quad u_y = -v_x.$$

$\Rightarrow$ This is the neccessary condn for diff. and analyticity of $f(z)$ at $z = z_0$

Exm :- $f(z) = \bar{z} = x - iy$ $u(x,y) = x$
$v(x,y) = -y$

$u_x = 1$, $u_y = 0$
$v_x = 0$, $v_y = -1$ But, $u_x \neq + v_y$ So, f is not diff.

Exm :- (Suppose C-R eqns satisfied) Does that imply differentiable ?

1. $f(x+iy) = \sqrt{|xy|}$ is not diff. at $(0,0)$ But, C-R eqn: are satisfied at $(0,0)$.

2. $$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

See Jain-Iyenger.

3. $$f(z) = \begin{cases} \dfrac{\bar{z}^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

$$u(x,y) = \begin{cases} \dfrac{x^3 - 3xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & \text{else} \end{cases}$$

$$v(x,y) = \begin{cases} \dfrac{y^3 - 3x^2y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & \text{else.} \end{cases}$$

Check $u_x = 1 = v_y$, $u_y = 0 = -v_x$. at $(0,0)$

But, f is not diff. at $z = (0,0)$

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{\bar{z}^2}{z^2}$$

Move along x-axis, the limit is 1

Move along $y = x$, the limit is -1.

Sufficient condition :-

Let, $f(z) = u(x,y) + iv(x,y)$ is defined in a nbd of (x_0, y_0). and

i) u_x, u_y, v_x, v_y exist in a nbd of (x_0, y_0)

ii) u_x, u_y, v_x, v_y are continuous at (x_0, y_0) and

$$u_x = v_y, \quad u_y = -v_x \quad \text{at } (x_0, y_0)$$

Then, $f'(z_0)$ exists and $f'(z_0) = u_x + iv_x$.

H.W. $f(z) = \frac{1}{z}, \quad \sin z, \quad e^z$.

In polar Co-ordinates :-

Let, $z = re^{i\theta}$. then

$$f(z) = u(x,y) + iv(x,y)$$
$$= U(r,\theta) + iV(r,\theta).$$

$x = r\cos\theta, \quad y = r\sin\theta$.

By chain Rule,

$$U_r = u_x x_r + u_y y_r = u_x \cos\theta + u_y \sin\theta$$

$$U_\theta = -u_x (r\sin\theta) + u_y (r\cos\theta)$$

$$V_r = v_x \cos\theta + v_y \sin\theta \qquad \left[\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right]$$
$$= -u_y \cos\theta + u_x \sin\theta$$
$$= -\frac{1}{r} U_\theta.$$

and
$$V_\theta = -v_x (r\sin\theta) + v_y (r\cos\theta)$$
$$= u_y (r\sin\theta) + u_x r\cos\theta$$
$$= r U_r.$$

So, C-R in ~~Et~~ Polar-coordinates :

$$V_r = -\frac{1}{r} U_\theta, \quad V_\theta = r U_r.$$

$$u_x = U_r \frac{\partial r}{\partial x} + U_\theta \frac{\partial \theta}{\partial x}$$

$$= U_r \cos\theta - \frac{\sin\theta}{r} U_\theta .$$

$r^2 = x^2 + y^2$

$r \frac{\partial r}{\partial x} = x$

$\therefore \frac{\partial r}{\partial x} = \cos\theta$

& $\quad V_x = \cos\theta \, V_r - \frac{\sin\theta}{r} V_\theta$

& $\tan\theta = \frac{y}{x}$

$\therefore \frac{\partial \theta}{\partial x} = -\frac{y}{r^2}$

$= -\frac{\sin\theta}{r}$

So, $\quad f'(z_0) = u_x + i v_x$

$$= \cos\theta (U_r + i V_r) - \frac{\sin\theta}{r} (U_\theta + i V_\theta)$$

$$= \cos\theta (U_r + i V_r) - \frac{\sin\theta}{r} (-r V_r + i r U_r)$$

[By C-R, $U_\theta = -r V_r$, $V_\theta = r U_r$]

$$= (\cos\theta - i \sin\theta) U_r + i (\cos\theta - i \sin\theta) V_r$$

$$\therefore \quad f'(z_0) = e^{-i\theta} (U_r + i V_r)$$

Also, $\quad f'(z_0) = \frac{1}{r} e^{-i\theta} (V_\theta - i U_\theta)$

Exm: $\quad f(z) = \sqrt[3]{r} \, e^{i\theta/3}$, $\quad r > 0$, $\alpha < \theta < \alpha + 2\pi$, $\alpha \in \mathbb{R}$.

$U(r,\theta) = \sqrt[3]{r} \cos\theta/3$, $\quad V(r,\theta) = r^{1/3} \sin\theta/3$.

$U_\theta = -r^{1/3} \frac{1}{3} \sin\theta/3$, $\quad V_r = \frac{1}{3} r^{-2/3} \sin\theta/3$

$\therefore U_\theta = -r V_r$.

$V_\theta = \frac{1}{3} r^{1/3} \cos\theta/3$, $\quad U_r = \frac{1}{3} r^{-2/3} \cos\theta/3$.

$\therefore V_\theta = r U_r$

$U_r, V_r, V_\theta, U_\theta$ are cont. So, f is diff. at all z. and

$$f'(z) = e^{-i\theta}\left(U_r + iV_r\right)$$

$$= e^{-i\theta} \frac{1}{3r^{2/3}}\left(\cos\theta/3 + i\sin\theta/3\right)$$

$$= \frac{1}{3\left(\sqrt[3]{r}\, e^{i\theta/3}\right)^2} = \frac{1}{3(f(z))^2}$$

H.W $\quad f(z) = z^n, \quad n \in \mathbb{N}.$

Prove that, $f'(z) = nz^{n-1}$

Let, $|f(z)| = K$, const. with f is analytic in D.

Then $f(z) = \text{const.}$ in D.

Ans: $\quad |f(z)|^2 = u^2 + v^2 = K^2$

$$\left.\begin{aligned} \text{So,}\quad & uu_x + vv_x = 0 \\ \& \quad & uu_y + vv_y = 0 \end{aligned}\right\}$$

$u_x = v_y, \quad v_x = -u_y$:

So, $uu_x - vu_y = 0$

& $uu_y + vu_x = 0$

So, $(u^2+v^2)u_x = 0$ & $(u^2+v^2)u_y = 0$

~~i.e.~~ If $u^2+v^2 = 0 = K^2$ then, $u = 0 = v$

So, $f(z) = 0$.

If not, then $u_x = 0 = u_y \Rightarrow u(x,y) = \text{Const.}$

By C-R, $v_x = 0 = v_y \Rightarrow v(x,y) = \text{Const.}$

So, $f(z) = u + iv = \text{Const.}$

Defn :- A fn $f(z)$ is said to be analytic at ∞ if $g(z) = f\left(\frac{1}{z}\right)$ is analytic at $z=0$.

~~Harmonic functions :-~~ **EXM:** $f(z) = \frac{1}{z}$

H.W. If $f(z)$ and $\bar{f}(z)$ are analytic on D, then f must be a constant fn.

Harmonic fn :- A real valued fn $H: \mathbb{R}^2 \to \mathbb{R}$, that has continuous 2nd order p.ds, is said to be harmonic in D if H satisfies the laplace eqn:

$$H_{xx} + H_{yy} = 0 \text{ . in } D.$$

EXM: Potential fns, $T(x,y) = e^{-y} \sin x$

$$\nabla^2 H = 0$$

$\boxed{}$ Let, $f(z) = u(x,y) + i v(x,y)$ is analytic in D.

Then, both u & v are harmonic.

If f is analytic in D, u & v have continuous 2nd order P.ds. So,

$$u_{xy} = u_{yx} \;\&\; v_{xy} = v_{yx}$$

f is analytic $\Rightarrow$ C-R eqns hold

i.e. $u_x = v_y$, $v_x = -u_y$

Now, $u_{xx} = v_{yx} = v_{xy} = -u_{yy}$

$\Rightarrow u_{xx} + u_{yy} = 0$

Ily, $v_{xx} + v_{yy} = 0$.

$\Rightarrow$ $v(x,y)$ is called the conjugate harmonic function of $u(x,y)$ in D.
(Harmonic Conjugate)

(Two harmonic fns u & v, that satisfy C-R eqns in D. Then, u & v constitute real & imag part of f. $f = u + iv$)

Theorem :- $f(z) = u(x,y) + iv(x,y)$ is analytic in D iff v is a harmonic conjugate of u.

Result :- The harmonic conjugate fn is unique except for an additive constant.

Let, (u, v_1) & (u, v_2) are two pairs.

$$\frac{\partial v_1}{\partial x} = -\frac{\partial u}{\partial y} = \frac{\partial v_2}{\partial x} \quad \& \quad \frac{\partial v_1}{\partial y} = \frac{\partial u}{\partial x} = \frac{\partial v_2}{\partial y}.$$

So, $$\frac{\partial}{\partial x}(v_1 - v_2) = 0 = \frac{\partial}{\partial y}(v_1 - v_2)$$

$\Rightarrow$ $v_1 - v_2 = K$, const.

$\therefore$ $\underline{v_1 = v_2 + K}$.

EXM :- $u(x,y) = x^2 - y^2$, $v(x,y) = 2xy$

$f(z) = z^2$ analytic in complex plane.

$\Rightarrow$ v is harmonic conjugate of u.

Is $g(z) = v + iu$ analytic ?

Converse of theorem may not hold!

$u(x,y) = \cancel{xy}\ 2xy, \quad v(x,y) = x^2 - y^2$

$f(z) = u + iv$ is not analytic.

Finding Harmonic Conjugate :- Type I

Find HC of $u(x,y) = y^3 - 3x^2y$.

i.e. to find $v(x,y)$ such that, v harmonic and $f(z) = u + iv$ is analytic.

$u_x = -6xy, \quad u_y = 3y^2 - 3x^2.$

$u_{xx} = -6y \quad u_{yy} = 6y.$

$\therefore u_{xx} + u_{yy} = 0$, hence harmonic.

By C-R, $v_y = u_x = -6xy$ & $v_x = -u_y = 3x^2 - 3y^2$

$\Rightarrow v(x,y) = -\int^x 6xy\, dy$

$= -3xy^2 + h(x)$

(Quick check $v_y \stackrel{?}{=} -6xy$)

$\therefore v_x = -3y^2 + \frac{dh}{dx} = 3x^2 - 3y^2$

$\Rightarrow \frac{dh}{dx} = 3x^2$

$\therefore h(x) = x^3 + c.$

So, $v(x,y) = -3xy^2 + x^3 + c.$

So, $f(z) = y^3 - 3x^2y + i(x^3 - 3xy^2 + c) = i(z^3 + c)$

H.W

Find HC. of $u(x,y) = x^2 - y^2 - y$.

Ans: $f(z) = \underline{z^2 + iz + ic}$

(By letting $y = 0$)

Type II

If $v(x,y)$ is HC of $u(x,y)$, ~~$e^y \sin x$~~ ~~$e^y \cos x$~~ with $v(x,y) = e^y \cos x$. Find u. & $f(z)$.

$$v_x = -e^y \sin x, \quad v_y = e^y \cos x$$

$$v_{xx} = -e^y \cos x, \quad v_{yy} = e^y \cos x$$

$$\therefore v_{xx} + v_{yy} = 0.$$

CR:
$$u_x = v_y = e^y \cos x, \quad u_y = -v_x = e^y \sin x.$$

$$\therefore u(x,y) = \int^{y} e^y \cos x \, dx$$

$$= e^y \sin x + h(y)$$

$$\therefore u_y = e^y \sin x + \frac{dh}{dy} = e^y \sin x$$

$$\therefore \frac{dh}{dy} = 0 \Rightarrow h(y) = K$$

$$\therefore u(x,y) = e^y \sin x + K.$$

$$\therefore f(z) = u + iv = e^y \sin x + K + i e^y \cos x$$

$$= e^y (\sin x + i \cos x) + K$$

$$= (\sin z + i \cos z + K) \quad [(\text{Check})]$$

Polar co-ordinates :-

$$f(z) = U(r,\theta) + i V(r,\theta)$$

C.R: $U_\theta = -r V_r$, $V_\theta = r U_r$

$U_{\theta r} = U_{r\theta}$ & $V_{\theta r} = V_{r\theta}$.

$$\Rightarrow \quad U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\theta\theta} = 0$$

$$\& \quad V_{rr} + \frac{1}{r} V_r + \frac{1}{r^2} V_{\theta\theta} = 0$$

Laplace eqn in polar - co-ordinate.