Elementary functions :-

1. Exponential f^n ✓
2. Trigonometric f^n ✓
3. Logarithmic f^n ✓
4. Hyperbolic f^n ✓

1. Exponential function:

We define e^z by

$$e^z = e^x(\cos y + i\sin y), \text{ which}$$

by Euler's formula,

$$e^z = e^x \cdot e^{iy}$$

$\Rightarrow$ One can see e^z as an extension of real f^n e^x such that

1) $e^z = e^x$ for $z = x$, real number
2) e^z is analytic for all $z \in \mathbb{C}$.
3) $(e^z)' = e^z$

$$\Rightarrow (e^z)' = (e^x \cos y)_x + i(e^x \sin y)_x$$
$$= e^z.$$

$\Rightarrow$ One can also define e^z as infinite series

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbb{C}.$$

Properties:

1) $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$

ii) $|e^z| = |e^x||e^{iy}| = e^x = e^{\operatorname{Re} z}$.

iii) $|e^z| = e^x \neq 0$, So, $e^z \neq 0$.

Entire fn which is never zero.

iv) $e^{z+2\pi i} = e^z \cdot e^{2\pi i} = e^z$.

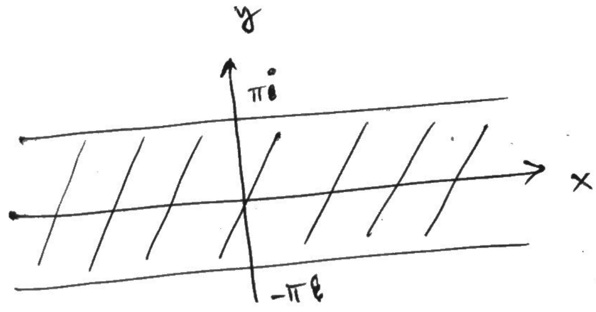
So, e^z is periodic with period $2\pi i$

v) $e^x > 0 \quad \forall x \in \mathbb{R}$.

But, $e^{i\pi} = \cos\pi + i\sin\pi = -1$

So, e^z can be negative.

$e^{i\pi/2} = i$, $e^{-i\pi/2} = -i$, $e^{-\pi i} = -1$



$-\pi < y \leq \pi$

All the values that e^z can assume are already assumed in the horizontal strip of width 2π.

This infinite strip is called a fundamental region of e^z.

Exm: Find $z = x+iy$ s.t. $e^z = 1+i$

$$e^x \cdot e^{iy} = re^{i\theta} = \sqrt{2}\, e^{i\pi/4}$$

So, $e^x = \sqrt{2}$, $y = \pi/4 \pm 2n\pi$, $n = 0, \pm1, \pm2, \dots$

& ~~$\ln x =$~~
$$x = \ln\sqrt{2} = \frac{1}{2}\ln 2$$

$$\therefore \quad z = \frac{1}{2}\ln 2 + i\left(2n+\frac{1}{4}\right)\pi, \quad n = 0, \pm1, \dots$$

Exm:

$$\left|e^{z^2}\right| \le e^{|z|^2}$$

$$\Rightarrow \quad \left|e^{z^2}\right| = e^{Re(z^2)}$$

$$\because \quad Re(z^2) \le |z|^2$$

So, $e^{Re(z^2)} \le e^{|z|^2}$.

2. Trigonometric fun:-

Euler's formula: $e^{ix} = \cos x + i\sin x$

$$e^{-ix} = \cos x - i\sin x$$

$$\Rightarrow \quad \cos x = \frac{e^{ix}+e^{-ix}}{2}, \quad \sin x = \frac{e^{ix}-e^{-ix}}{2i}$$

So, we define as an extension:

$$\cos z = \frac{e^{iz}+e^{-iz}}{2} \quad \& \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}.$$

Properties :-

i) $\frac{d}{dz}\sin z = \cos z$, $\frac{d}{dz}\cos z = -\sin z$.

ii) $\tan z = \frac{\sin z}{\cos z}$, $\sec z = \frac{1}{\cos z}$, $\operatorname{cosec} z = \frac{1}{\sin z}$

iii) $\sin z = \sin x \cos(iy) + \cos x \sin(iy)$

$\sin(iy) = i\sinh y$, $\cos(iy) = \cosh y$

$\therefore \sin z = \sin x \cosh y + i \sinh y \cos x$.

$\cos z = \cos x \cosh y - i \sin x \sinh y$.

4. Hyperbolic fn :-

The hyperbolic fns are defined by

$\cosh z = \frac{e^z + e^{-z}}{2}$, $\sinh z = \frac{e^z - e^{-z}}{2}$

Properties:

i) $(\cosh z)' = \sinh z$, $(\sinh z)' = \cosh z$

ii) $\cosh(iz) = \cos z$, $\sinh(iz) = i\sin z$

$\cos(iz) = \cosh z$, $\sin(iz) = i\sinh z$.

iii) $\sinh(z_1+z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$

$\cosh(z_1+z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

3. <u>Logarithmic function :-</u>

Suppose we want to solve

$$e^{\omega} = z \quad \text{for some nonzero } z \in \mathbb{C}.$$

Note, $e^{\omega} \neq 0$, i.e. $z \neq 0$.

Let, $z = re^{i\theta}$, $r>0, -\pi < \theta \leq \pi$, and $\omega = u + iv$.

then $$e^{\omega} = e^{u} \cdot e^{iv} = re^{i\theta}.$$

i.e. $r = e^{u}$ and $v = \theta + 2n\pi$, $n \in \mathbb{Z}$.

So, $u = \ln r$ (Real natural logarithm)

Therefore, $\omega = \ln r + i(\theta + 2n\pi)$, $n = 0, \pm 1, \pm 2, \ldots$

So, we write

$$\omega = \ln z := \ln r + i(\theta + 2\pi n), \quad \text{so that} \quad \cdots (*)$$

$$e^{\ln z} = z. \quad (z \neq 0).$$

From $(*)$, it is clear that, $\ln z$ is multivalued and defined by:

$$\underline{\ln z = \ln|z| + i(\theta + 2n\pi)}, \quad n = 0, \pm 1, \pm 2, \ldots \quad (z \neq 0)$$

$$\Rightarrow \quad \ln z = \ln|z| + i \arg z.$$

▨ The principal value of $\ln z$ is the value obtained from $(*)$, when $n=0$, and denoted by $\operatorname{Ln} z$.

So, $\operatorname{Ln} z = \ln|z| + i \operatorname{Arg} z$.

$\Rightarrow$ $\text{Ln } z$ is well-defined and single valued. for $z \neq$

and

$$\ln z = \text{Ln } z + 2n\pi i \quad , \quad n = 0, \pm 1, \pm 2, \ldots$$

$\Rightarrow$ For real $z = re^{i0} > 0$,

$$\text{Ln } z = \ln r + i0 \quad (\text{Arg } z = 0)$$

$$\therefore \quad \text{Ln } r = \ln r.$$

$\Rightarrow$ For real and negative ~~real~~, $z = r < 0$

$$\text{Arg } z = \pi.$$

So, $\text{Ln } z = \ln |z| + \pi i$.

$\Rightarrow$ $$\ln(e^z) = \ln |e^z| + i \arg(e^z)$$

Let, $z = x + iy$, then,

$$|e^z| = e^x.$$

$$\& \arg(e^z) = y + 2n\pi, \quad n \in \mathbb{Z}$$

So,
$$\ln(e^z) = \ln(e^x) + i(y + 2n\pi)$$
$$= (x + iy) + i2n\pi$$
$$= z + 2n\pi i, \quad n \in \mathbb{Z}.$$

Exm :-

$$\ln 1 = 2n\pi i, \quad n \in \mathbb{Z}, \quad \text{Ln } 1 = 0.$$

$$\ln(-1) = \ln 1 + i(\pi + 2n\pi) = (2n+1)\pi i, \quad n \in \mathbb{Z}$$

$$\text{Ln}(-1) = \pi i$$

$$\ln i = i\frac{\pi}{2} + 2n\pi i, \quad \text{Ln } i = \frac{i\pi}{2}.$$

Properties :-

1. $\ln(z_1 z_2) = \ln z_1 + \ln z_2$

Not true for principal values:

$$z_1 = -1 = z_2$$

$$\operatorname{Ln} z_1 z_2 = \operatorname{Ln} 1 = 0$$

$$\operatorname{Ln} z_1 = i\pi = \operatorname{Ln} z_2$$

2. $\ln\left(z_1/z_2\right) = \ln z_1 - \ln z_2.$

3. $z = re^{i\theta}$ is a non-zero complex., then

$$\ln z = \ln r + i\theta$$

and $\theta = \operatorname{Arg} z + 2n\pi$, $n \in \mathbb{Z}$.

If $r > 0$ and $\alpha < \theta < 2\pi + \alpha$ for $\alpha \in \mathbb{R}$,

then, $\ln z = \ln r + i\theta$ gives a single valued fn. with $u(r,\theta) = \ln r$, $v(r,\theta) = \theta$.

C-R. eqn: $\quad r u_r = v_\theta$, $u_\theta = -r v_r$. Satisfied.

The fn $\ln z = \ln r + i\theta$ is continuous as well as analytic in the domain: $r > 0$, $\alpha < \theta < \alpha + 2\pi$.

Also,

$$(\ln z)' = e^{-i\theta}(u_r + i v_r) = \frac{1}{re^{i\theta}} = \frac{1}{z},$$

for $|z| > 0$, $\alpha < \arg z < \alpha + 2\pi$.

$\Rightarrow$ A branch of a multi-valued fn $f(z)$ is any single-valued fn F that is analytic for $z \in D$ such that $F(z)$ is one of the values of $f(z)$, $z \in D$.

$\Rightarrow$ A branch cut is a portion of a line or curve that is introduced in order to define a branch F.

$\Rightarrow$ Any point that is common to all branch cuts of f is called a branch point.

Exm: The origin and the ray $\theta = \alpha$ make up the branch cut for the above branch of ~~Log z~~ $\ln z$.

Exm: $\text{Log}\, i^3 = \text{Log}(-i) = -i\frac{\pi}{2}$.

$3\,\text{Log}\, i = 3 \cdot i\frac{\pi}{2} = i\frac{3\pi}{2}$.

So, ~~$\text{Log}\, i^3 \neq 3\,\text{Log}\, i$~~. $\text{Ln}\, i^3 \neq 3\,\text{Ln}\, i$

General Powers:

$z \neq 0$. $c \in \mathbb{C}$. z^c is defined by

$$z^c = e^{c \ln z}.$$

As, $\ln z$ is multivalued, z^c is multivalued.

Exm: $i^i = e^{i \ln i} = e^{i(i(\frac{\pi}{2} + 2n\pi))}, \quad n \in \mathbb{Z}$

$= e^{-\frac{\pi}{2} - 2n\pi}, \quad n \in \mathbb{Z}$.

$\Rightarrow$ $\text{Log}(z) / \text{Ln}(z)$ is not continuous in $\mathbb{C}$.

Let, $z = -x$, a neg. real no.

$z_{\pm} = -x \pm i\epsilon, \quad \epsilon > 0$.

Now, $\lim_{\epsilon \to 0} z_{\pm} = -x$. But,

$$\lim_{\epsilon \to 0} \text{Ln}(z_{\pm}) = \ln(x) \pm i\pi.$$

So, $\text{Ln}(z)$ is not cont. on the neg. real line.