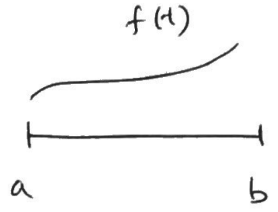
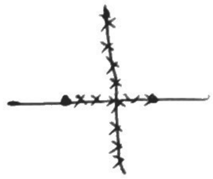
Complex Integration :—

H-W $\ln \cancel{\log}(z^2-1)$. Find branch & cuts.

$$\int_a^b f(t)\,dt$$





If $f : \mathbb{R} \to \mathbb{C}$ i.e. $f(t) = u(t) + iv(t)$, then

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt,$$ provided

u and v are integrable.

\# Let, $f : \mathbb{R} \to \mathbb{C}$ is integrable then P.t.

$$\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt.$$

Let, $\int_a^b f(t)\,dt = Re^{i\theta}$ where

$$R = \left|\int_a^b f(t)\,dt\right| \quad \cdots (I)$$

Now, $R = \int_a^b e^{-i\theta} f(t)\,dt = \int_a^b (U+iV)\,dt$, say

then, $$R = \int_a^b U(t)\,dt \quad \cdots (II)$$

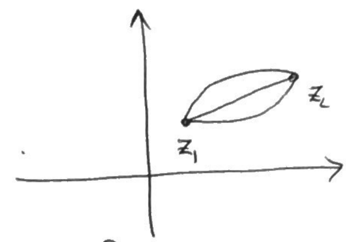
But, $U(t) = \mathrm{Re}\left(e^{-i\theta} f(t)\right) \le \left|e^{-i\theta} f(t)\right| = |f(t)|$

So, $$\int_a^b U(t)\,dt \le \int_a^b |f(t)|\,dt \quad \cdots (III)$$

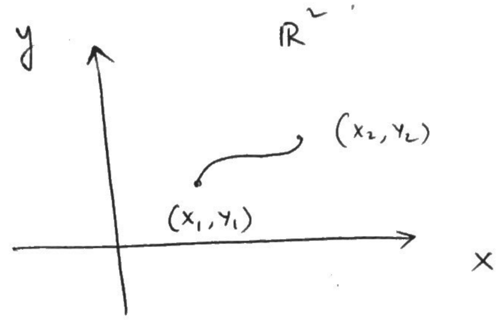
(I), (II) & (III) $\Rightarrow$ $$\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt.$$

⇒ Complex integration of complex-valued f^n:

$$\int_{z_1}^{z_2} f(z)\,dz \,.$$



⇒ Need to introduce the notion of path!



⇒ $C: \gamma: [a,b] \to \mathbb{C}$ s.t. $\gamma(a) = (x_1, y_1)$ and piecewise diff.

$\gamma(b) = (x_2, y_2)$

So, parametrically, $\gamma(t) = (x(t), y(t))$, $t \in [a,b]$.

$x(t)$ & $y(t)$ are real valued continuous f^n.

⇒ A curve $C: \gamma(t)$ is closed if $\gamma(a) = \gamma(b)$

⇒ C is simple ~~Jordan~~ if it it doesn't intersect itself, except at initial and end pts.

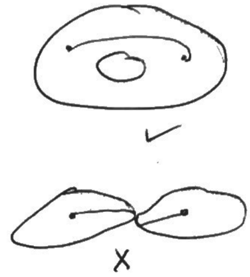
Exm: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. $\quad x = a\cos\theta$, $y = b\sin\theta$ $\quad 0 \le \theta \le 2\pi$.

⇒ C is said to be smooth if $\dfrac{d\gamma}{dt}$ is continuous in $[a,b]$ with $\dfrac{d\gamma}{dt} \neq 0$.

⇒ A simple closed curve γ is called a Jordan curve.

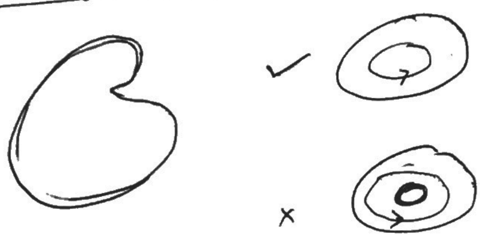
Line Integral :-

Connected Region :



~~can~~ can be joined by a smooth curve.

Simply Connected :-



any ~~simply~~ simple closed curve can be shrunk to a pt.

Positive Orientation :-

"Counter-clockwise".

More-precisely, γ is said to be +vely oriented if a person walking on γ, always has the domain to his left.

$\gamma_1 : \; \gamma_1(t) = e^{it}, \; 0 \le t \le 2\pi$ +ve

$\gamma_2 : \; \gamma_2(t) = e^{-it}, \; 0 \le t \le 2\pi$ -ve

$\gamma_3 : \; \gamma_3(t) = -e^{+it}$ +ve

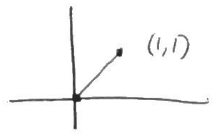
$\gamma_4 : \; \gamma_4(t) = -e^{-it}$ -ve.

Def^n :- Let, C be a piecewise smooth curve joining z_1 and z_2 in the complex plane, given by $\gamma(t)$, $a \le t \le b$. Let, f be a complex valued continuous f^n along C. Then, the line integral of f along C is defined as:

$$\int_{z_1}^{z_2} f(z)\,dz = \int_C f(z)\,dz := \int_a^b f(\gamma(t))\,\gamma'(t)\,dt.$$

Exm: $f(z) = x^2 + iy^2$, $C : \gamma(t) = (t, t)$ $\quad 0 \le t \le 1$

Find $\int_C f(z)\,dz$.

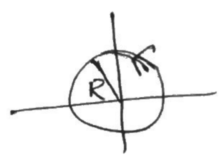


$$\int_C f(z)\,dz = \int_0^1 f(\gamma(t))\,\gamma'(t)\,dt$$

$$= \int_0^1 (t^2 + it^2)(1+i)\,dt$$

$$= \int_0^1 2it^2\,dt = \frac{2}{3} i.$$

Exm:- $f(z) = \frac{1}{z}$, $C : \gamma(t) = Re^{it}$, $0 \le t \le 2\pi$, $R \neq 0$



$$\int_C f(z)\,dz = \int_0^{2\pi} f(Re^{it})\, iRe^{it}\,dt$$

$$= \int_0^{2\pi} \frac{1}{R} e^{-it}\, iRe^{it}\,dt$$

$$= 2\pi i. \quad \text{(Independent of } R\text{)}$$

Exm:- $f(z) = 1$, C is any smooth curve.

$$\int_C f(z)\,dz = \int_a^b \gamma'(t)\,dt = \gamma(b) - \gamma(a)$$

which is indep. of curve C.

$\Rightarrow$ If C is a closed path, then the line integral of f is denoted by

$$\oint_C f(z)\,dz.$$

Fundamental theorem of Integral Calculus (Line Integral)

Suppose f is the derivative of an analytic function F, ie. $F'(z) = f(z)$ and F is analytic on the smooth curve C. Then,

$$\int_C f(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

Proof :- Let, C is parametrized as $\gamma(t)$. $a \le t \le b$.

Let, $g(t) = F(\gamma(t))$, $a \le t \le b$. Then

$$g : [a,b] \to \mathbb{C}.$$

Now,
$$g'(t) = F'(\gamma(t)) \cdot \gamma'(t) = f(\gamma(t))\,\gamma'(t).$$

So,
$$\int_C f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt = \int_a^b g'(t)\,dt = g(b) - g(a) = \underline{F(\gamma(b)) - F(\gamma(a))}.$$

Cor :- If C is a closed contour in D and $f: D \to \mathbb{C}$ has an anti-derivative in D, then,

$$\oint_C f(z)\,dz = 0.$$

ML - inequality :-

Suppose $f(z)$ is continuous on a contour c having length L with $|f(z)| \leq M$ on c.

Then,

$$\left| \int_c f(z)\,dz \right| \leq ML.$$

Proof :- Let, c is parametrized by

$$\gamma(t), \quad a \leq t \leq b.$$

Now,
$$\left| \int_c f(z)\,dz \right| = \left| \int_a^b f(\gamma(t))\,\gamma'(t)\,dt \right|$$
$$\leq \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt$$
$$\leq M \int_a^b |\gamma'(t)|\,dt$$

Now, let, $\gamma(t) = (x(t), y(t)) \quad a \leq t \leq b.$

Then,
$$\int_a^b |\gamma'(t)|\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$$
$$= L, \text{ arc length}.$$

So, $$\left| \int_c f(z)\,dz \right| \leq ML$$

ExM : Find an upper bound of $\left| \int_c \frac{dz}{z^2+10} \right|$, $C : \gamma(t) = 2e^{it}$, $t \in [-\pi, \pi]$

$$|z^2+10| \geq 10 - |z|^2$$
$$= 10 - 4 = 6$$

$\therefore |f(z)| \leq \frac{1}{6}$ along c.

Also, L = length of $c = 4\pi$.

$$\therefore \left| \int_c f(z)\,dz \right| \leq ML = \frac{4\pi}{6} = \frac{2\pi}{3}.$$

H.W Exm: $\left| \int_C \frac{e^z}{z+1}\, dz \right| \le 4\pi e^2$.

Exm:- Compute $\int_{|z|=r} x\, dz$

$x = \frac{z+\bar z}{2}$. & $|z|^2 = r^2$

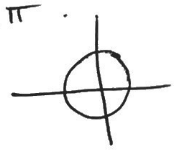
$\Rightarrow z\bar z = r^2 \Rightarrow \bar z = r^2/z$

$\Rightarrow \quad x = \frac{z + r^2/z}{2}$.

So, $\int_{|z|=r} x\, dz = \frac{1}{2}\int_{|z|=r} \left(z + r^2/z\right) dz = \frac{1}{2} r^2 \cdot 2\pi i = \underline{\pi r^2 i}$.

H.W Compute $\int_{|z|=R} \bar z^n\, dz$, $n = 0, \pm 1, \dots$

Exm:- $\int_{|z|=1} \sqrt{z}\, dz$ $\quad C: \gamma(t) = e^{it}$, $-\pi \le t \le \pi$.



$\sqrt{z} = z^{1/2} = e^{\frac{1}{2}\ln z}$.

Let us consider the Principal value of $\sqrt{z}$.

i.e. $\sqrt{z} = e^{\frac{1}{2}\mathrm{Ln}(z)}$.

Now, $f(\gamma(t)) = e^{\frac{1}{2}\mathrm{Ln}(e^{it})} = e^{\frac{1}{2}it}$, $-\pi < t \le \pi$.

and $f(\gamma(t)) = e^{\frac{1}{2}i\pi}$ at $t = -\pi$.

$$\int_{|z|=1} \sqrt{z}\, dz = \int_{-\pi}^{\pi} f(\gamma(t))\, \gamma'(t)\, dt$$

$$= \int_{-\pi}^{\pi} e^{it/2}\, i e^{it}\, dt = i\int_{-\pi}^{\pi} e^{i3t/2}\, dt = \frac{2}{3} i^2 \left(e^{i\frac{3\pi}{2}} - e^{-i\frac{3\pi}{2}}\right)$$

$$= \frac{4}{3} i.$$