Cauchy Integral Theorem :-

Defn : (Entire function) : A complex valued fn $f : \mathbb{C} \to \mathbb{C}$ is said to be entire if f is analytic in $\mathbb{C}$.

Exm : e^z, $\sin(z)$, $\cos(z)$

* **Theorem :-** If f is analytic in a simply connected domain D, then

$$\int_C f(z)\,dz = 0$$

for every simple closed contour C in D.

(Weak Version)

If f is analytic in a simply connected domain D with f' continuous on D, then, $\int_C f(z)\,dz = 0$ for every closed contour C in D.

Proof :-

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy)$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

$$\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right] \underline{CR}$$

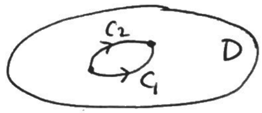
$$\overset{\text{Green's}}{=} \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$= 0 \quad [\text{By C-R eq}^{\text{n}}]$$

Cor: Let, C_1 & C_2 be any contours in a simply connected domain D with same initial & terminal pts. Then,

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

for any analytic $f\underset{..}{n}$ f in D.



Hint: Consider $C = C_1 - C_2$

then, by Cauchy Integral thm,

$$\int_{C} f(z)\,dz = 0$$

$$\Rightarrow \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

$$\therefore \int_{C} f(z)\,dz = \int_{C_2} f(z)\,dz$$

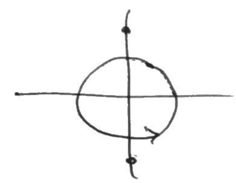
This is the path independence property of line integrals.

ExM:- $$\int_{|z|=1} \frac{dz}{z^2+4} = \int_{|z|=1} \frac{dz}{(z+2i)(z-2i)}$$

f is analytic inside the simply connected domain $\{z \mid |z| \leq 1\}$. So, by Cauchy Integral thm,

$$\int_{|z|=1} \frac{dz}{z^2+4} = 0.$$

Exm :
$$\int_{|z|=2} \frac{z e^{z}}{(z^2+9)^2}\, dz$$



$$= \int_{|z|=2} \frac{z e^{z}}{(z+3i)(z-3i)}\, dz$$

$$= 0 \qquad [\text{By CIT}]$$

* Exm :-
$$\int_{|z|=1} \bar{z}\, dz = \int_0^{2\pi} e^{-it}\, i e^{it}\, dt$$

$$= 2\pi i \neq 0.$$

Because $f(z) = \bar{z}$ is not analytic in $\{z \mid |z| \le 1\}$

* Exm :-
$$\int_{|z|=1} \frac{dz}{z^2} = 0$$
but, $f(z) = \frac{1}{z^2}$ is not analytic at $z=0$. So, CIT is suff. not necc. (Converse of CIT is not true)

H.W.
$$\int_{|z|=1} \frac{z+1}{z^2+2z}\, dz \quad , \quad \int_{|z|=1} \frac{dz}{z} = 2\pi i$$

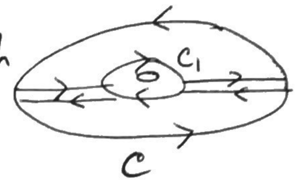
Cauchy Integral Theorem for multiply connected domains

Let, D is not simply connected. D will be called multiply connected.

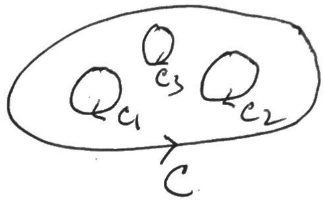
1) Let, C be a simple closed curve ~~in the~~ +vely oriented.

II) C_k, $k = 1, 2, \dots, n$ are simple closed curves interior to C, and all -vely oriented and these are disjoint and interiors of C_k have no common pts.

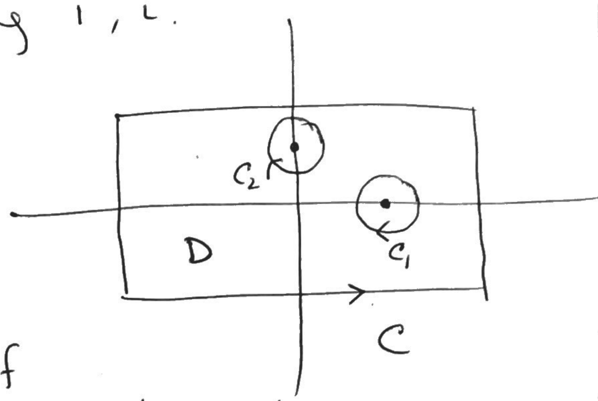
If f is analytic on all contours C, C_K's and ~~pts~~ in pts inside C and exterior to each C_K, then,



$$\int_C f(z)\,dz + \sum_{K=1}^{n} \int_{C_K} f(z)\,dz = 0$$



Exm :- $\displaystyle\int_C \frac{dz}{(z-1)(z-i)}$ where C is a closed rectangle containing $1, i$.



$f(z) = \dfrac{1}{(z-1)(z-i)}$ is not analytic at $z=1, i$. Now, consider C_1 & C_2, two circles of radius r_1 & r_2 such that C_1 & C_2 does not intersect and C_1, C_2 lie inside C. Then D is multiply connected domain. By Cauchy Integral thm,

$$\int_C \frac{dz}{(z-1)(z-i)} = -\int_{C_1} \frac{dz}{(z-1)(z-i)} - \int_{C_2} \frac{dz}{(z-1)(z-i)}$$

$$= \frac{1}{1-i} \int_{C_1} \left[\frac{1}{z-i} - \frac{1}{z-1}\right] dz + \frac{1}{1-i} \int_{C_2} \left[\frac{1}{z-i} - \frac{1}{z-1}\right] dz$$

Now, C_1 encloses 1, C_2 encloses i.

~~But~~ By CIT, $\displaystyle\int_{C_1} \frac{dz}{z-i} = 0 = \int_{C_2} \frac{dz}{z-1}$

$-C_1: z_1(t) = 1 + r_1 e^{it}$, $-C_2: z_2(t) = i + r_2 e^{it}$, $t \in [0, 2\pi]$

$$\int_{-C_1} \frac{dz}{z-1} = \int_0^{2\pi} \frac{1}{\cancel{r_1}} e^{-it} \cancel{r_1} i e^{it}\, dt = 2\pi i = \int_{-C_2} \frac{dz}{z-i}$$

So,

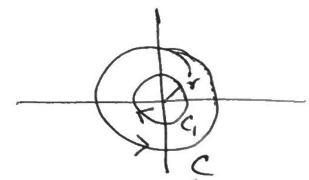
$$\int_C \frac{dz}{(z-1)(z-i)} = -\frac{1}{1-i}(-2\pi i) + \frac{1}{1-i}(-2\pi i)$$

$$= 0.$$

<u>H.W</u>

$$I = \int_{|z|=1} \frac{dz}{z^3 + 3iz^2} = \frac{1}{9}\int_C \frac{dz}{z} - \frac{i}{3}\int_C \frac{dz}{z^2} - \frac{1}{9}\int_C \frac{dz}{z+3i}$$

Not analytic at $z=0$.



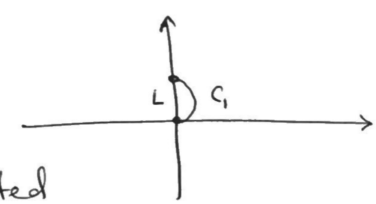
CIT (Extend) $\Rightarrow$

$$I = -\frac{1}{9}\int_{C_1} \frac{dz}{z} + \frac{i}{3}\int \frac{dz}{z^2}$$

$$= -\frac{1}{9}(-2\pi i) = \frac{2\pi i}{9}.$$

<u>H.W.</u>

$$\int_0^i \sinh(\pi z)\, dz$$



Consider any ~~simply~~ simply connected domain D, containing $0, i$, and the line segment L from 0 to i.

~~By~~ $f(z)$ is analytic in any such domain D, so

$$\int_{C_1} f(z)\,dz = \int_L f(z)\,dz = \int_0^i f(z)\,dz.$$

$L: \quad \gamma(t) = it, \quad 0 \le t \le 1$

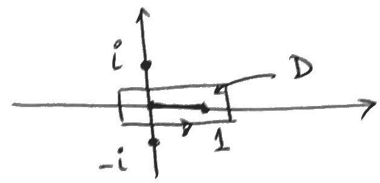
So, $$\int_0^i f(z)\,dz = \int_L f(z)\,dz = \int_0^1 \sinh(i\pi t)\, i\, dt$$

$$= -\int_0^1 \sin(\pi t)\,dt$$

$$= \frac{1}{\pi}\cos(\pi t)\Big]_0^1$$

$$= -\frac{2}{\pi}.$$

H.W. $\displaystyle\int_0^1 \frac{\tan^{-1} z}{1+z^2}\, dz$



Consider domain D. Inside D, $f(z) = \dfrac{\tan^{-1} z}{1+z^2}$ is analytic & D is simply connected.

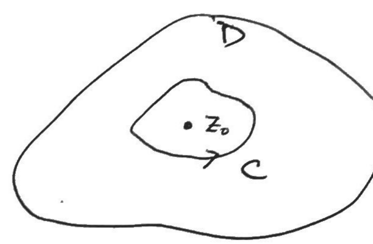
L: Line joining 0 & 1, $\gamma(t) = t$, $0 \le t \le 1$

$$\int_0^1 \frac{\tan^{-1} z}{1+z^2}\, dz = \int_L \frac{\tan^{-1} z}{1+z^2}\, dz = \int_0^1 \frac{\tan^{-1} t}{1+t^2}\, dt = \frac{1}{2}\left(\frac{\pi}{4}\right)^2.$$

Cauchy Integral Formula (CIF) :-

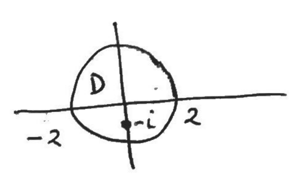
Let, $f(z)$ be analytic in a simply connected domain D, containing the positively oriented simple closed contour C. If z_0 is any point inside C, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz$$



Exm : $\displaystyle\int_{|z|=R} \frac{dz}{z - z_0} = 2\pi i$

Exm : $\displaystyle\int_{|z|=2} \frac{z}{(z+i)(9-z^2)}\, dz$



$f(z) = \dfrac{z}{9-z^2}$ is analytic in D. Then,

$$\int_{|z|=2} \frac{f(z)}{(z+i)}\, dz = 2\pi i\, f(-i) = 2\pi i \frac{-i}{10}$$
$$= \frac{\pi}{5}.$$

Exm :-

$$\int_{|z|=1} \frac{dz}{3-\bar{z}}$$

Along $|z|=1$. $\quad 3-\bar{z} = 3 - \frac{|z|^2}{z} = 3 - \frac{1}{z}$

$$= \frac{3z-1}{z}$$

$$\therefore \int_{|z|=1} \frac{dz}{3-\bar{z}} = \frac{1}{3}\int_{|z|=1} \frac{z\,dz}{z-1/3}$$

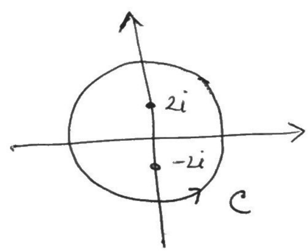
$f(z) = z$ is analytic in unit disk, D. So, by CIF, $\left(\frac{1}{3} \in D\right)$

$$\frac{1}{3}\int_{|z|=1} \frac{z\,dz}{z-1/3} = \frac{1}{3}\, 2\pi i\, f(1/3)$$

$$= \frac{1}{3}\, 2\pi i\, \frac{1}{3} = \frac{2\pi i}{9}.$$

Exm :

$$\int_C \frac{dz}{z^2+4} \qquad C: |z| = 4.$$



$$= \frac{1}{4i}\int_C \frac{dz}{z-2i} - \frac{1}{4i}\int_C \frac{dz}{z+2i}$$

Consider D as the ~~unit~~ disk $\{z : |z| \leq 4\}$ and $f(z) = 1$. f is analytic in D. ~~So, by~~ And $\pm 2i \in D$.

By CIF,

$$\int_C \frac{dz}{z^2+4} = \frac{1}{4i}\left[2\pi i - 2\pi i\right] = 0.$$

H.W.

$$\int_{|z-1|=1} \frac{z^2+1}{z^2-1}\, dz \qquad f(z) = \frac{z^2+1}{z+1} \text{ analytic in } D = \{z : |z-1| \leq 1\}$$

Generalized Cauchy Integral Formula :-

Let, $f(z)$ be analytic in a simply connected domain D, containing a simple closed contour C. Then, $f(z)$ has derivatives of all order ~~at z_0~~ in D and for a pt. z_0 inside C,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad n=1,2,\dots$$

$\Rightarrow$ f is analytic at a pt $z_0 \Rightarrow$ f has derivatives at z_0 of infinite order.

Exm :-

$$\int_{|z|=1} \frac{e^{5z}}{z^3}\,dz = \frac{2\pi i}{2!} f^{(2)}(0)$$

$$= 25\pi i$$

Hint :-

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$$

$$= \lim_{h\to 0} \frac{1}{2\pi i h}\left[\int_C \left[\frac{f(z)}{z-z_0-h} - \frac{f(z)}{z-z_0}\right] dz\right]$$

$$= \lim_{h\to 0} \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)(z-z_0-h)}\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz$$

Cauchy Inequality :- If $C : |z-z_0| = R$. then,

$$|f^{(n)}(z_0)| = \left|\frac{n!}{2\pi i}\right| \left|\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right| \le \frac{n!}{2\pi}\frac{M}{R^{n+1}}\,2\pi R$$

$$= \frac{n!\,M}{R^n}.$$