Power Series :-

Sequence :-

$$f : \mathbb{N} \to \mathbb{C}.$$

$$f(n) = a_n + i b_n = z_n, \quad a_n, b_n \in \mathbb{R}.$$

Exm: $\{f(n)\}_n = \left\{ \frac{1}{n} + \frac{i}{n^2} \right\}_n$

$$1+i, \ \frac{1}{2} + \frac{i}{4}, \ \frac{1}{3} + \frac{i}{3^2}, \ \ldots$$

Exm: Real ~~Sqn~~ Sequence:

$$\left\{ \frac{1}{n} \right\}_n \quad \{a_n\}_n. \quad f : \mathbb{N} \to \mathbb{R}.$$

Convergence:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \to 0. \text{ as } n \to \infty$$

A seq$^{\underline{n}}$ $\{f(n)\}_n$ is said to be convergent, if for any and fix $\epsilon > 0$, $\exists \ N_0 \in \mathbb{N}$ s.t.

$$|f(n) - l| < \epsilon \quad \text{when. } n \geq N_0.$$

i.e. $|\underbrace{a_n + i b_n}_{z_n} - (l_1 + i l_2)| < \epsilon$ when, $n \geq N_0$.

\# ~~a~~ l is called the limit of the ~~sq~~ sequence $\{f(n)\}_n$.

\# $\{z_n\}_n$ is converging to $l_1 + i l_2$ iff $\{a_n\}_n \to l_1$ & $\{b_n\}_n \to l_2$, where $z_n = a_n + i b_n$.

Exm $\quad z_n = \frac{1}{n^3} + i$ converges to i.

$$= a_n + i b_n$$

$$\lim_{n\to\infty} \frac{1}{n^3} = 0, \qquad \lim_{n\to\infty} b_n = 1.$$

So, $\displaystyle\lim_{n\to\infty} z_n = i$

NOT True in polar co-ordinates.

$$z_n = -2 + i\,\frac{(-1)^n}{n^2} = r_n e^{\theta_n}, \qquad \theta_n = \mathrm{Arg}(z_n).$$

$$\lim_{n\to\infty} z_n = -2$$

$$r_n = |z_n| = \sqrt{4 + \frac{1}{n^4}} \to 2 \quad \text{as } n\to\infty$$

$$\theta_n = \tan^{-1}\left(\frac{(-1)^n/n^2}{-2}\right) \to \begin{cases} \pi, & n = \text{even} \\ -\pi, & n = \text{odd}. \end{cases}$$

Series: $\{z_n\}_n$ be a sequence.

$$z_1 + z_2 + z_3 + \cdots = \sum_{n=1}^{\infty} z_n$$

Let, $S_N = \sum_{n=1}^{N} z_n = z_1 + \cdots + z_N \qquad N = 1, 2, \ldots$

So, $\{S_n\}_n$ be a seq$^{\underline{n}}$ corresponding to $\sum z_n$.

It is called the seq$^{\underline{n}}$ of partial sums.

$\Rightarrow$ $\sum z_n$ is convergent if $\{S_n\}_n$ is convergent and the limit S is called the sum of the series.

$$S = \sum_{n=1}^{\infty} z_n$$

A necessary condition for the series $\sum_{n=1}^{\infty} z_n$ to be convergent is $z_n \to 0$ as $n \to \infty$.

Absolute Convergence :-

$\sum z_n$ is absolutely conv. if $\sum |z_n|$ is convergent.

$$\sum |z_n| = |z_1| + |z_2| + \cdots$$

Comparison Test :- Consider $\sum_{n=1}^{\infty} z_n$ and a known series $\sum_{n=1}^{\infty} \omega_n$ and let, $|z_n| \le \omega_n$ $\forall n \geq N_0$ another $\sum_{n=1}^{\infty} v_n$ and let, $v_n \le |z_n|$ $\forall n \geq N_0$. then.

1) if $\sum \omega_n$ converges, so does $\sum z_n$

11) if $\sum v_n$ diverges, so does $\sum z_n$.

Comparison Test :- Consider two real series $\sum a_n$ & $\sum b_n$ with $0 \le a_n \le b_n$ $\forall n \geq N_0$.

1) if $\sum b_n$ converges, so does $\sum a_n$

11) if $\sum a_n$ diverges, so does $\sum b_n$.

Absolute convergence $\Rightarrow$ Convergence.

$\sum |z_n| = \sum \sqrt{a_n^2 + b_n^2}$ convergent.

$0 \le |a_n| \le \sqrt{a_n^2 + b_n^2}$ & $0 \le |b_n| \le \sqrt{a_n^2 + b_n^2}$

By comparison test, $\sum |a_n|$ & $\sum |b_n|$ converge.

So, $\sum a_n$, $\sum b_n$ converge. $\Rightarrow \sum z_n$ converges.

$\sum \frac{1}{n^p}$ converges if $p > 1$.

$\sum z^n = 1 + z + z^2 + \cdots = \frac{1}{1-z}$ for $\underline{|z|<1}$.

$$S_N(z) = \sum_{n=1}^{N} z^n = \frac{1 - z^{N+1}}{1-z} \rightarrow \frac{1}{1-z} \quad \text{as } N \to \infty$$

$\underline{|z|<1}$.

Geometric series.

Power Series :-

$$\sum_{n=1}^{\infty} f_n = f_1 + f_2 + \cdots \qquad \text{Constant series}$$

$$\sum_{n=1}^{\infty} f_n(z) = f_1(z) + f_2(z) + \cdots \qquad z \in \mathbb{C}.$$

~~Sequence of fn.~~

Series of fn..

Let, $z_0 \in \mathbb{C}$.

$$\left.\begin{aligned} \sum_{n=1}^{\infty} f_n(z_0) &\equiv \sum_{n=1}^{\infty} g_n^0 \\ \sum_{n=1}^{\infty} f_n(z_i) &= \sum_{n=1}^{\infty} g_n^i \end{aligned}\right\} \text{may or may not converge.}$$

Let, for $z \in D$, $\sum f_n(z)$ converges to $f(z)$, i.e.

$$f(z) = \sum_{n=1}^{\infty} f_n(z), \quad z \in D.$$

Uniform convergence :-

For every $\epsilon > 0$, $\exists$ $N_0 \in \mathbb{N}$, independent of z, s.t.

$$|S_n(z) - f(z)| < \epsilon \quad \forall n \geq N_0 \quad \forall z \in D$$

where, $S_n(z) = \sum_{k=1}^{n} f_n(z)$.

EXM. $f_n(z) = \frac{1}{nz}$, $z \neq 0$

$$\lim_{n\to\infty} f_n(z) = 0.$$

$\left|\frac{1}{nz}\right| < \epsilon$ if $n > \frac{1}{\epsilon|z|} = N_0$.

Converges pointwise., not uniformly.

Let, $\epsilon_0 < |z| < 1$. Then.

$\left|\frac{1}{nz}\right| < \epsilon$ if $n > \frac{1}{|z|\epsilon} > \frac{1}{\epsilon\epsilon_0}$ indep of z.

$\{f_n(z)\}$ converges uniformly in $\epsilon_0 < |z| < 1$.

Defn :- An infinite series of the form
$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$ is called
a power series about $z = z_0$, $a_n \in \mathbb{C}$.

$\Rightarrow$ $f_n(z) = a_n(z - z_0)^n$.

$\Rightarrow$ If $z_0 = 0$: $\sum_{n=0}^{\infty} a_n z^n$ is a power series in powers of z.

$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$

The power series converges for $z = 0$ &
$$a_0 = \sum_{n=0}^{\infty} a_n z^n \Big|_{z=0}$$

What about $z \neq 0$?

EXM $\sum_{n=0}^{\infty} z^n = 1 + z + \cdots$ Geometric series

Converges for $|z| < 1$.

Convergence of Power Series :-

Consider a power series: $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, $a_n \in \mathbb{C}$.

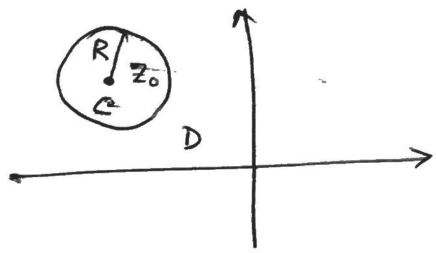
Then, one of the following is true:

i) the PS converges only at $z = z_0$.

ii) the PS converges for $|z-z_0| < R$ and diverges for $|z-z_0| > R$. The series may or may not converge at points on $|z-z_0| = R$.

iii) the PS converges in $\mathbb{C}$.

$\Rightarrow$ In case of (ii), the real number R is called the radius of convergence.



Calculation of R :-

A. Ratio test :-

Let, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$

a) if $L = 0$, $R = \infty \Rightarrow$ P.S. converges in $\mathbb{C}$.

b) if $L = \infty$, $R = 0 \Rightarrow$ P.S. " only at $z = z_0$

c) $R = \frac{1}{L}$.

B. Root test :-

Let, $\lim_{n\to\infty} |a_n|^{1/n} = L$. $\left(\limsup_{n\to\infty} |a_n|^{1/n}\right)$

then, $R = \frac{1}{L}$.

Exm:-

$$\sum_{n=0}^{\infty} \left(\frac{n+2}{3n+1}\right)^n (z-4)^n$$

Root test: $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{n+2}{3n+1} = \frac{1}{3}$.

$\therefore$ $\underline{R = 3}$

Exm. $\sum \frac{1}{n!} z^n$.

Ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{(n+1)} = 0$.

$\therefore$ $R = \infty$: P.S. converges for all $z \in \mathbb{C}$.

Exm. $\sum_{n=0}^{\infty} \frac{1}{(3+i)^n} z^{3n}$

Let, $z^3 = \omega$ .. $\sum_{n=0}^{\infty} \frac{1}{(3+i)^n} \omega^n$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1}{3+i}\right| = \frac{1}{\sqrt{10}}.$$

$\therefore$ $R = \sqrt{10}$.

So, the series $\sum a_n \omega^n$ converges for $|\omega| < \sqrt{10}$.

& diverges for $|\omega| > \sqrt{10}$.

i.e. converges for $\underline{|z| < 10^{1/6}}$, & div. for $\underline{|z| > 10^{1/6}}$.

H.W. $\sum \frac{z^{3n}}{4^n n^\alpha}$, $\underline{\alpha > 0}$

$$\lim_{n\to\infty} \left|\frac{z^{3(n+1)}\, 4^n n^\alpha}{4^{n+1} (n+1)^\alpha z^{3n}}\right|$$

$= \frac{|z^3|}{4} < 1$ i.e. $\underline{|z| < 4^{1/3}}$

$\sum_{n=0}^{\infty} u_n$

If $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = L < 1$, then $\sum u_n$ converges, else diverges.

$\Rightarrow$ Let, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z|<R$. , $\underline{R>0}$

i.e. $f(z)$ is represented as a power series, centered at $z=0$.

<u>Termwise derivative</u>

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = f'(z) \quad , \ |z|<R.$$

The derived series of a P.S. has the same radius of convergence as the original P.S.

<u>Termwise Integration</u>

The P.S $\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}$ & has the same radius of convergence as the original P.S.

<u>Exm</u>: $\sum \frac{(n!)^2 z^n}{(2n)!} = \sum u_n$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{[(n+1)!]^2 z^{n+1} (2n)!}{(2n+2)!\,(n!)^2 z^n}\right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^2 |z|}{2(n+1)(2n+1)} = \frac{|z|}{4}$$

The P.S. converges if $\frac{|z|}{4} < 1$ i.e. $|z|<4. = R$.

$$\sum \frac{n(n!)^2 z^{n-1}}{(2n)!} = \sum v_n$$

$$\lim_{n\to\infty} \left|\frac{v_{n+1}}{v_n}\right| = \lim_{n\to\infty} \frac{(n+1)}{n} \cdot \frac{(n+1)}{2(2n+1)} |z| = \frac{|z|}{4}.$$

So, $R=4$.

Also, $\sum \frac{(n!)^2}{(2n)!\,(n+1)} z^{n+1}$ has Radius of conv. $=4$.

$\sum a_n (z-z_0)^n = f(z)$ represents an analytic fn for $|z-z_0| < R$ if $R > 0$.

The derivatives of the sum fn is obtained by differentiating the P.S. term by term. and f is ~~and~~ infinitely differentiable.

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)\, a_n (z-z_0)^{n-k}.$$

for $|z-z_0| < R$.

$\Rightarrow$ Sum of a P.S is analytic and has derivatives of all orders, which are also analytic fn.

In fact, ~~the~~ converse is also true!

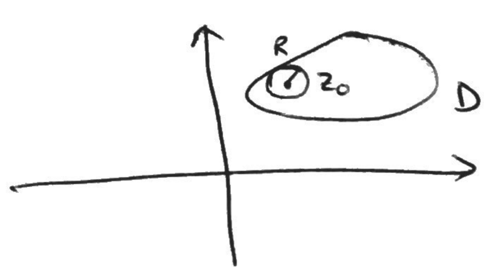
<u>Taylor's Theorem</u> :-

Let, $f(z)$ be analytic in D, and $z_0 \in D$. Then $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{where}$$

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \quad n = 0, 1, 2, \dots$$

This representation is valid in the largest open disk with center z_0 in which $f(z)$ is analytic



Exm. $f(z) = e^z$ is an entire fn.

$f^{(n)}(0) = 1. \quad \forall n.$

So, $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad |z| < \infty$

$z_0 = 0$: Maclaurin Series.

Exm. $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$

$$= \frac{1}{2i}\left[\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!}\right]$$

$$= \frac{1}{2i} \sum_{n=0}^{\infty} \{1 - (-1)^n\} \frac{i^n z^n}{n!}$$

$$= \frac{1}{2i} \sum_{n=0}^{\infty} \frac{2 i^{2n+1} z^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \quad |z| < \infty.$$

Term by term diff:

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \quad |z| < \infty$$

Exm. $\sinh(z) = -i \sin(iz) = -i \sum_{n=0}^{\infty} (-1)^n \frac{(iz)^{2n+1}}{(2n+1)!}$

$$= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \quad |z| < \infty$$

$$\Rightarrow e^{iy} = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!}$$

$$= \cos(y) + i \sin(y)$$ [Euler's Formula]

EXM.

$$Ln(1+z) = \sum_{n=0}^{\infty} a_n z^n$$

with $a_n = \dfrac{f^{(n)}(0)}{n!}$

$a_0 = 0, \quad a_1 = \dfrac{1}{1+z}\Big|_{z=0} = 1, \quad a_2 = \dfrac{-1}{2!(1+z)^2}\Big|_{z=0}$

$= -\dfrac{1}{2}.$

$a_n = \dfrac{(-1)^{n-1}}{n}.$

$$\therefore \; Ln(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}$$

with $R = \lim_{n\to\infty} \left| \dfrac{a_n}{a_{n+1}} \right| = 1.$

So, $Ln(1+z) = z - \dfrac{z^2}{2} + \dfrac{z^3}{3} - \cdots \quad , |z|<1$

EXM. $\dfrac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z|<1$

EXM. $f(z) = \dfrac{1}{(z+1)(z+2)}$

$$= \frac{1}{z+1} - \frac{1}{z+2}$$

$$= (1+z)^{-1} - \frac{1}{2}(1+z/2)^{-1}$$

$$= \sum_{n=0}^{\infty} (-1)^n z^n - \frac{1}{2} \sum_{n=0}^{\infty} (z/2)^n (-1)^n \text{ for } |z|<1$$

$$= \sum_{n=0}^{\infty} \left((-1)^n - \frac{(-1)^n}{2^{n+1}} \right) z^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(1 - \frac{1}{2^{n+1}} \right) z^n. \quad , |z|<1$$

EXM. $\dfrac{\sin z}{\cos z} = \left(\sum_{n=0}^{\infty} (-1)^n \dfrac{z^{2n+1}}{(2n+1)!} \right) \cdot \left\{ 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \cdots \right\}^{-1}$

$$= \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \right)\left(1 + \frac{z^2}{2} + \frac{5z^4}{24} \cdots \right)$$

$$= z + \left(\frac{1}{2} - \frac{1}{3!} \right) z^3 + \cdots$$

$\cos z = 0 \Rightarrow z = \dfrac{n\pi}{2}. \quad n = \pm 1, \pm 2, \ldots$

$R = \pi/2.$

Laurent Series :-

→ What if $f(z)$ is not analytic at z_0?

If f is analytic at z_0, then,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{for some nbd of } z_0.$$

We generalize Taylor's series, and call it Laurent series.

Consider the series:

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z-z_0)^n} \quad \cdots (1)$$

or

$$\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$

The region of convergence of $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ may be $\mathbb{C}$ or $|z-z_0|<R$. Inside the region of convergence, the P.S. converges to an analytic fn. $f_1(z)$, i.e.

$$f_1(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad |z-z_0|<R.$$

Let, $\omega = \frac{1}{z-z_0}$. then, $\sum_{n=1}^{\infty} b_n \frac{1}{(z-z_0)^n} = \sum_{n=1}^{\infty} b_n \omega^n$.

Let, the 2nd P.S. converges to some analytic fn., i.e.

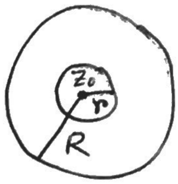
$$\phi(\omega) = \sum_{n=1}^{\infty} b_n \omega^n \quad \text{for } |\omega| < 1/r.$$

i.e. $f_2(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ for $\underline{|z-z_0|>r}$

If $r < R$, then, $f_1(z)$ & $f_2(z)$ are both analytic in the annulus $r < |z-z_0| < R$. Hence,

$$f(z) = f_1(z) + f_2(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}.$$

$$= \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$



where, $c_n = \begin{cases} a_n, & n \geq 0 \\ b_{-n}, & n < 0. \end{cases}$

This expression is called a Laurent series about z_0.

$\Rightarrow$ If $r > R$, the Laurent series diverges everywhere.

$\Rightarrow$ If $r = R$, the LS, diverges everywhere, except possibly at pts. on $|z-z_0| = R$.

EXM. $\displaystyle\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{z^n}{n^2}$ conv. for $|z| = 1$.

$\displaystyle\sum_{n=-\infty}^{\infty} z^n$ div. for $|z| = 1$

$\displaystyle\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{z^n}{n}$ conv. everywhere on $|z| = 1$, except at $z = 1$.

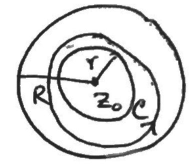
Laurent's Theorem :-

Suppose $f(z)$ is analytic in the annulus $r < |z-z_0| < R$. Then the representation

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$ is valid throughout the annulus. The co-eff. are given by:

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz \quad , \quad n = 0, \pm 1, \pm 2, \dots$$

where C is any +vely oriented simple closed contour in the annulus about z_0.



Remark: i) $a_n \neq \dfrac{f^{(n)}(z_0)}{n!}$ as Taylor's, since f is not analytic in $|z-z_0| \leq R$.

~~Exm~~ ii) $\displaystyle\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ is called the Principle part of the Laurent series. & $\displaystyle\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is called the analytic part.

Exm. $\cancel{\frac{1}{(z-1)}}$ $f(z) = 1/(1-z)$ about $z = 0$.

$f(z)$ is not analytic at $z = 1$. We consider two regions: i) $|z| < 1$ & ii) $|z| > 1$.

f is analytic in $|z| < 1$:

$$f(z) = \sum_{n=0}^{\infty} z^n \quad , \quad |z| < 1$$

For $|z|>1$:

$$\frac{1}{1-z} = -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} \qquad \left|\frac{1}{z}\right|<1$$

$$= -\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$$

$$= -\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) \quad \text{for } |z|>1$$

This is a Laurent Series. ($\rho=1$, $R=\infty$)

Exm. $f(z) = \dfrac{\sin z}{z^2}$. $|z|>0$.

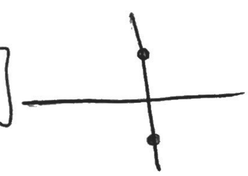
$$= \frac{1}{z^2}\left(\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}\right)$$

$$= \frac{1}{z^2}\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)$$

$$= \frac{1}{z}-\frac{z}{3!}+\frac{z^3}{5!}-\cdots, \quad |z|>0$$

Exm. $f(z) = \dfrac{1}{z^2+1} = 2i\left[\dfrac{1}{z-i}-\dfrac{1}{z+i}\right]$

$$= 2i\left[-\frac{1}{i}\left(1-\frac{z}{i}\right)^{-1}-\frac{1}{i}\left(1+\frac{z}{i}\right)^{-1}\right]$$



$$= -2\sum_{n=0}^{\infty}\left(\frac{z}{i}\right)^n - 2\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{i}\right)^n$$

$$= -2\sum_{n=0}^{\infty}\frac{[1-(-1)^n]}{i^n}\, z^n \quad \text{for } |z|<1$$

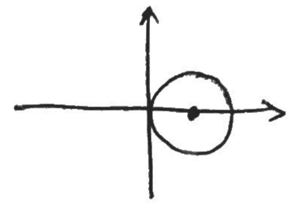
Laurent Series about $z=i$:

$$f(z) = \frac{1}{z-i}\,\frac{1}{z+i} = \frac{1}{z-i}\,\frac{1}{2i}\left(1+\frac{z-i}{2i}\right)^{-1}$$

$$= \frac{1}{2i}\,\frac{1}{z-i}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2i)^n}(z-i)^n$$

$$= -\sum_{n=-1}^{\infty}\left(\frac{i}{2}\right)^{n+2}(z-i)^n \quad \text{for } 0<|z-i|<2.$$

Laurent series of $\frac{1}{z-2}$ in $|z-1|>1$



$$\frac{1}{z-2} = \frac{1}{(z-1)-1}$$

$$= \frac{1}{z-1} \cdot \frac{1}{1-\frac{1}{(z-1)}}$$

$$= \frac{1}{(z-1)}\left(1-\frac{1}{(z-1)}\right)^{-1}, \quad \left|\frac{1}{z-1}\right|<1$$

$$= \frac{1}{(z-1)}\left(1+\frac{1}{z-1}+\frac{1}{(z-1)^2}+\frac{1}{(z-1)^3}+\cdots\right)$$

$$= \frac{1}{(z-1)}+\frac{1}{(z-1)^2}+\frac{1}{(z-1)^3}+\cdots$$

$$= \sum_{n=1}^{\infty}\frac{1}{(z-1)^n}. \quad \text{in } \underline{|z-1|>1}.$$

$\Rightarrow$ The Laurent series of a given analytic fn $f(z)$ in its annulus of convergence is unique.

Laurent series of $\frac{1}{z-2}$ in $|z|>2$.

$$\frac{1}{z-2} = \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$$

$$= \frac{1}{z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\cdots\right)$$

$$= \frac{1}{z}+\frac{2}{z^2}+\frac{2^2}{z^3}+\cdots$$

$$= \underline{\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}}. \quad \text{for } \underline{|z|>2}.$$