~~Singularity & Zero of f~~

Branches of Logarithm :-

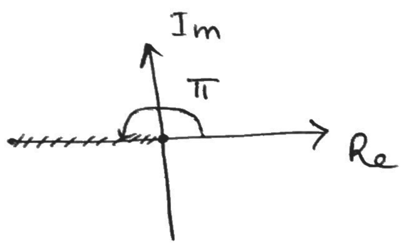
$\Rightarrow$ $f(z) = \ln(z)$ is a multi-valued fn, $f : \mathbb{C} - \{0\} \to \mathbb{C}$

$\Rightarrow$ $\mathrm{Ln}_\alpha(z)$, $\alpha \in \mathbb{R}$ and $\mathrm{Ln}(z)$ are single-valued, but not continuous, since.

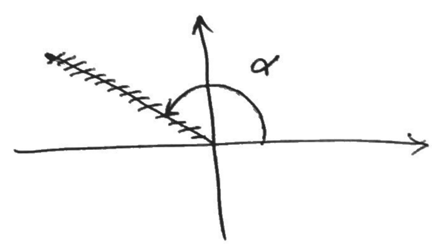
$$\mathrm{Ln}(z) = \ln|z| + i\,\mathrm{Arg}(z) \quad , \quad \mathrm{Arg}\, z \in (-\pi, \pi]$$

or
$$\mathrm{Ln}_\alpha(z) = \ln|z| + i\,\mathrm{Arg}_\alpha(z) \quad , \quad \mathrm{Arg}_\alpha(z) \in [\alpha, 2\pi + \alpha)$$

are not continious along $\theta = \pi$ and $\theta = \alpha$ respectively.



Ln(z)



$\mathrm{Ln}_\alpha(z)$

$\Rightarrow$ If we remove $\theta = \pi$ (for $\mathrm{Ln}(z)$) & $\theta = \alpha$ (for $\mathrm{Ln}_\alpha(z)$) line segment, from their respective domain, the restricted fns

$$\mathrm{Ln} : \mathbb{C} - \{(x, 0) : x \le 0\} \to \mathbb{C}$$

$$\& \quad \mathrm{Ln}_\alpha : \mathbb{C} - \{(r, \theta) : r \geq 0,\ \theta = \alpha\} \to \mathbb{C}.$$

become analytic. <u>Branches</u>

$\Rightarrow$ So, $F_\alpha(z) = \mathrm{Ln}_\alpha(z)$, $\alpha \in \mathbb{R}$ are all branches of $f(z) = \ln(z)$. We can identify the <u>principal branch</u> $\mathrm{Ln}(z)$ as

$$\mathrm{Ln}(z) = \mathrm{Ln}_{-\pi}(z) = F_{-\pi}(z)$$

$\Rightarrow$ So, there are infinitely many branches of $\ln(z)$ and they are of the form

$$F_\alpha(z) = Ln_\alpha(z), \quad \alpha \in \mathbb{R}.$$

Where, $Ln_\alpha(z) = \ln|z| + i\theta, \quad \theta \in (\alpha, 2\pi+\alpha)$,

the line segment $\theta = \alpha$ is removed to make ~~the~~ $Ln_\alpha(z)$ analytic.

$\Rightarrow$ For $\alpha = -\pi$, we get the principal branch.

$\Rightarrow$ Among all $F_\alpha(z) = Ln_\alpha(z)$, $Ln_0(z)$ and $Ln_{-\pi}(z)$ $\left(= Ln(z)\right)$ are easier to find.

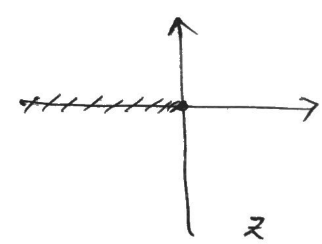
$\Rightarrow$ Comparison of $\ln(z)$ & $\ln(g(z))$ for some ~~$f(z)$~~. $g(z)$.

$\Rightarrow$ Similarly, $Ln_\alpha(g(z))$ are all its branches for $\alpha \in \mathbb{R}$.

To find branch pt & branch cut:

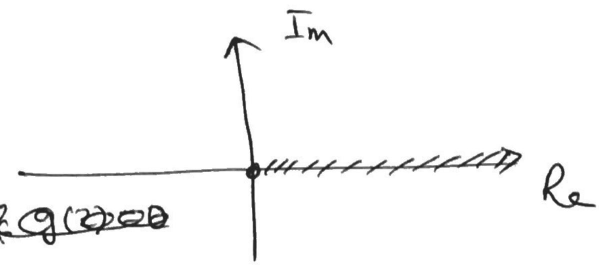
$Ln_0(z)$ & $Ln(g(z))$ (Principal branch)

Branch pt : $z = 0$ & $g(z) = 0$.



Branch cut : $Re(z) < 0$ & $Im(z) = 0$ | $Re(g(z)) < 0$ & $Im(g(z)) = 0$

$\underline{Ln_0(z)}$ & $\underline{Ln_0(g(z))}$.



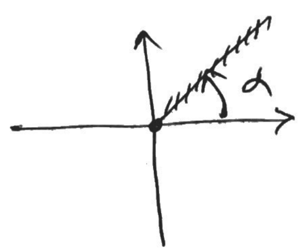
z - plane

<u>Branch pt:</u> $z = 0$ | $g(z) = 0$

<u>Branch cut:</u>

$Re(z) > 0$, & $Im(z) = 0$ | $Re(g(z)) > 0$ & $Im(g(z)) = 0$.

$Ln_\alpha(z)$ & $Ln_\alpha(g(z))$



<u>Branch pt</u>

$z = 0$ | $g(z) = 0$

<u>Branch cut</u>

~~Arg~~ $\arg(z) = \alpha$ | $\arg(g(z)) = \alpha$.

<u>Problems :-</u>

1. Find a branch of $\ln(z^2 + 2iz + 2)$, which is analytic at $z = i$.

<u>Soln :-</u> All branches of $f(z) = \ln(g(z))$ are of the form $Ln_\alpha(g(z))$, $\alpha \in \mathbb{R}$.

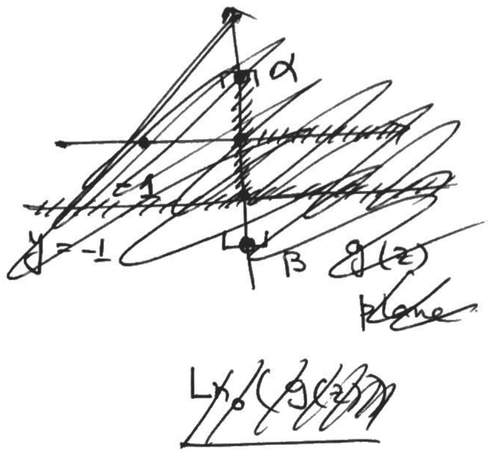
We need a particular branch, amongst them, which is analytic at $z = i$.

That is to say, we need $Ln_\alpha(g(z))$ for some $\alpha \in \mathbb{R}$ such that, Ln_α is analytic at $g(i) = -1$.

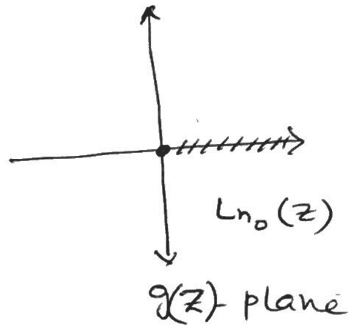
$\Rightarrow$ This will ~~happen~~ not happen for $\alpha \in \mathbb{R}$.

$\Rightarrow$ The easiest possible branches are $Ln_0(g(z))$ & $Ln_{-\pi}(g(z))$. $\left(= Ln(g(z))\right)$.

But, $Ln_{-\pi}(g(z))$ is not analytic at -1.



$\Rightarrow$ So, we go for $Ln_0(g(z))$.



Branch pt :-

$$g(z) = 0$$

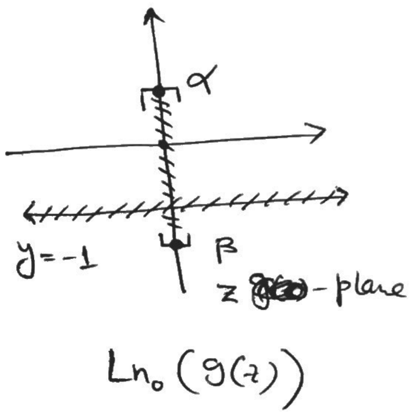
$$\therefore z^2 + 2iz + 2 = 0$$

$$\Rightarrow z = (\sqrt{3}-1)i, -(\sqrt{3}+1)i$$

$$= \alpha, \beta$$

Branch cuts :-

$Re(g(z)) > 0$ & $Im(g(z)) = 0$



$z = x + iy$:

$x^2 - y^2 - 2y + 2 > 0$ & $xy + x = 0$

$\Rightarrow x(y+1) = 0$

$x = 0$ or $y = -1$

$y = -1 \Rightarrow x^2 + 3 > 0$

which is always true for $x \in \mathbb{R}$.

So, $y = -1$ is a part of the branch-cut.

$x = 0 \Rightarrow y^2 + 2y - 2 < 0$

$$\Rightarrow \left(y - \frac{\alpha}{i}\right)\left(y - \frac{\beta}{i}\right) < 0$$

$$\therefore \beta/i < y < \alpha/i$$

So, $\left[\frac{\beta}{i}, \frac{\alpha}{i}\right]$ is another part of the branch cut.

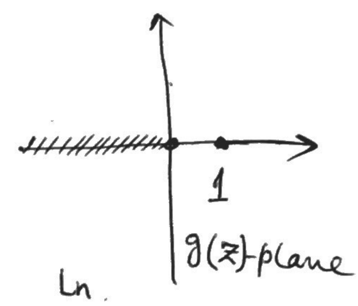
The total union of the two is the branch-cut for $\mathrm{Ln}_0(g(z))$.

2. Find a branch of $\ln(z^2+1)$, which is analytic at $z=0$.

All its branches are $\mathrm{Ln}_\alpha(z^2+1)$, $\alpha \in \mathbb{R}$.

Among these, we need to find a branch, which is analytic at $z=0$, i.e. Ln_α analytic at $g(0)=1$.

So, we take the principal branch $\mathrm{Ln}(z^2+1)$ of $\ln(z^2+1)$.

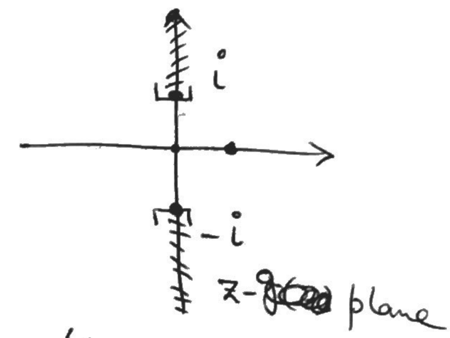


Branch pt:

$g(z)=0$

$\Rightarrow z^2+1=0 \quad \Rightarrow z=\pm i$

Branch cut:



$\mathrm{Re}(g(z))<0$ & $\mathrm{Im}(g(z))=0$

$\Rightarrow x^2-y^2+1<0$ & $xy=0$

$\Rightarrow \underline{x=0}$ or $\underline{y=0}$

$x=0$: $y^2-1>0$

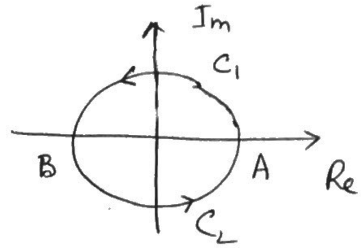
$\Rightarrow |y|>1$

$y=0$: $x^2+1<0$, which is never true.

So, the branch cut is $L=\{(0,y): |y|\geq 1\}$

Therefore, $\mathrm{Ln}(z^2+1)$, defined on $\mathbb{C}-L$, is a branch of $\ln(z^2+1)$, that is analytic at $z=0$.

$$\int_{|z|=1} z^{1/2}\, dz$$



where $f(z) = z^{1/2} = e^{\frac{1}{2}\mathrm{Ln}(z)}$

$= e^{\frac{1}{2}(\ln|z| + i\,\mathrm{Arg}(z))}$

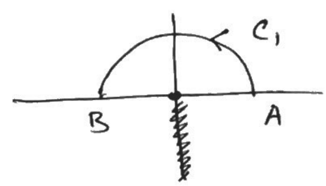
$= e^{\frac{1}{2} i\,\mathrm{Arg}(z)} = e^{\frac{1}{2} i\,\mathrm{Arg}(e^{i\theta})}$ on $\underline{|z|=1}$.

$= e^{i\theta/2}$

where, $\mathrm{Arg}(z) \in (-\pi, \pi]$, $\theta \in [-\pi, \pi]$

$f(z)$ is piecewise continuous on C_1, so the integral exists. But, the principal branch of $z^{1/2}$ is not defined on $\theta = \pi$, i.e. at B.

Consider another branch,



$$f_1(z) = e^{\frac{i}{2}\theta} \quad \text{where,}$$

$$-\frac{\pi}{2} < \theta < \frac{3\pi}{2}.$$

$f_1(z)$ is analytic on C_1, and

$$f_1(z) = f(z) \quad \text{on } C_1 \text{ except at B.}$$

$f_1(z)$ has an anti-derivative

$$F_1(z) = \frac{2}{3} z^{3/2} = \frac{2}{3} e^{\frac{3}{2} i\theta},$$

$$-\pi/2 < \theta < \frac{3\pi}{2}. \quad \text{and}$$

$F_1(z)$ is analytic.

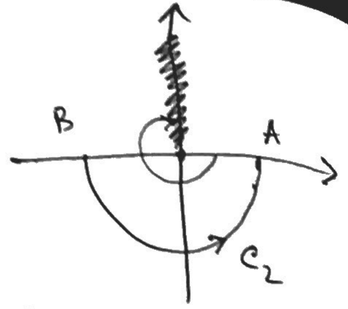
So, by Fundamental thm of Calculus,

$$\int_C z^{1/2}\, dz = \int_A^B f_1(z)\, dz = F_1(z)\Big]_A^B$$

$$= \frac{2}{3} e^{\frac{3}{2} i\theta}\Big]_0^{\pi} = \frac{2}{3}\left[-i - 1\right]$$

Now, consider the branch

$$f_2(z) = e^{\frac{1}{2}i\theta}, \quad \theta \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right).$$



Then, $f_2(z) = f(z)$ on C_2, except at B.

But, $f_2(z)$ is analytic in C_2 and

$$F_2(z) = \frac{2}{3} z^{3/2} = \frac{2}{3} e^{\frac{3}{2}i\theta}, \quad \theta \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right)$$

is analytic on C_2. So,

$$\int_{C_2} f(z)\, dz = \int_B^A f_2(z)\, dz$$

$$= F_2(z)\Big]_B^A$$

$$= \frac{2}{3} e^{i\frac{3}{2}\theta}\Big]_{-\pi}^{0}$$

$$= \frac{2}{3}\left[1 - i\right]$$

So, $\displaystyle\int_C f(z)\, dz = -\frac{4}{3} i$.

Take the branch $f(z) = z^{1/2} = e^{\frac{i\theta}{2}}, \quad \theta \in (0, 2\pi).$

Then, $\displaystyle\int_{C_1} f(z)\, dz = \frac{2}{3} e^{i\frac{3}{2}\theta}\Big]_0^{\pi}$ $\quad \left(F_1(z) = \frac{2}{3} e^{i\frac{3}{2}\theta}, \ \theta \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)\right)$

& $\displaystyle\int_{C_2} f(z)\, dz = \frac{2}{3} e^{i\frac{3}{2}\theta}\Big]_{\pi}^{2\pi}$ $\quad \left(F_2(z) = \frac{2}{3} e^{i\frac{3}{2}\theta}, \ \theta \in \left(\frac{\pi}{2}, \frac{5\pi}{2}\right)\right)$

$$\therefore \int_C f(z)\, dz = \frac{2}{3}\left[e^{i3\pi} - 1\right]$$

$$= -\frac{4}{3}.$$