Zeros, Singularities & Residue Theorem :-

Singular Point :- A point $z_0 \in \mathbb{C}$ is said to be a singular point of a fn $f: D \to \mathbb{C}$ if f is not analytic at z_0, but every neighbourhood of z_0 contains a point at which f is analytic.

Exm. $f(z) = \frac{1}{z}$: $z = 0$ is a singular pt.

$f(z) = \dfrac{z+1}{z^3(z^2+1)}$: $0, \pm i$ are three singular points

$f(z) = |z|^2$: ~~f is not analytic at z=0, or any non-z~~

f is no-where analytic. So, it has no singular pt.

$f(z) = \bar{z}$: f is nowhere analytic. So, it has no singular point.

Isolated Singularity :- A singular point z_0 of f is said to be isolated if there is a deleted neighbourhood $0 < |z - z_0| < \delta$ of z_0 $\left(N_\delta(z_0) - \{z_0\} \right)$ throughout which f is analytic.

Exm $f(z) = \tan z$, $\pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots$ are isolated singularities.

$f(z) = \mathrm{Ln}(z)$, principal branch. $z = 0$ is a singularity. But not an isolated singularity.

$\Rightarrow$ If f has isolated singularity at $z=z_0$, by Laurent theorem, we can write,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

for $0 < |z-z_0| < R$, for some $R>0$

$\Rightarrow$ $\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ is called the principal part of the L.S.

Now, if $b_K = 0$ $\forall K \geq N_0$, i.e. the principal part has finite terms

$$\frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_{N_0}}{(z-z_0)^{N_0}},$$

then $z=z_0$ is called a pole of f, and N_0 is called its order.

If $N_0 = 1$, z_0 is called a simple pole.

<u>Defn</u> :- A function f, having an isolated singular point z_0, has <u>a pole at z_0 of order</u> $m (\in \mathbb{N})$ if

$$\lim_{z \to z_0} (z-z_0)^m f(z) \; \cancel{\text{exists}} \; = l \, (\neq 0, \infty)$$

$\Rightarrow$ ~~If~~ ~~z_0 is not~~

If an isolated singular point z_0 of f is not a pole, it is called an <u>essential singular point</u>.

EXM. $f(z) = \frac{1}{z(z-2)^5} + \frac{1}{(z-2)^2}$

$z = 0, 2$ are two isolated singular points of f.

Now, $\lim_{z \to 0} z f(z) = -\frac{1}{2^5} \neq 0$: z is a simple pole

$\lim_{z \to 2} (z-2)^5 f(z) = \frac{1}{2} \neq 0$: $z = 2$ is a pole of order 5.

EXM $f(z) = e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n}$, $0 < |z|$

The Principal part of the L.S. has infinite terms,

so, $z = 0$ is an essential singular point.

~~Residues of~~

Removable Singularity :- f is said to have a removable singularity at $z = z_0$ if $f(z)$ is not analytic at z_0, but can be made analytic by assigning a suitable value $f(z_0)$.

EXM. $f(z) = \frac{\sin z}{z}$, $z \neq 0$

f has removable singularity at $z=0$, since $f(0) = 1$ makes f analytic.

Residues :-

Let, f has isolated singularity at $z = z_0$.

Then, by Laurent Theorem, we can write:

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n, \quad \text{with} \quad 0 < |z - z_0| < R,$$

where

$$C_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz,$$

C is any +vely oriented simple closed curve in $0 < |z - z_0| < R$.

Now $n = -1$.

$$C_{-1} = \frac{1}{2\pi i} \int_C f(z) \, dz$$

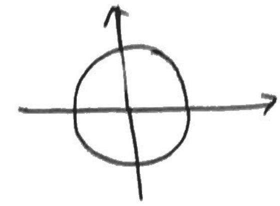
i.e. $\int_C f(z) \, dz = 2\pi i \, C_{-1}$

C_{-1} is the co-eff. of the term $\frac{1}{(z - z_0)}$.

Define $C_{-1} = \operatorname{Res}_{z = z_0} f(z)$, residue of f at the isolated singular point z_0.

and $$\int_C f(z) \, dz = 2\pi i \operatorname{Res}_{z = z_0} f(z)$$

EXM. $\displaystyle\int_{|z|=1} z^2 \sin\left(\frac{1}{z}\right) dz$



$z=0$ is an isolated singular point of

$f(z) = z^2 \sin\left(\frac{1}{z}\right)$.

Now, in $0 < |z| < \infty$,

$$f(z) = z^2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!\, z^{2n+1}}$$

$$= z^2\left(\frac{1}{z} - \frac{1}{3!\, z^3} + \frac{1}{5!\, z^5} - \cdots\right)$$

$$= z - \frac{1}{3!\, z} + \frac{1}{5!\, z^3} - \cdots$$

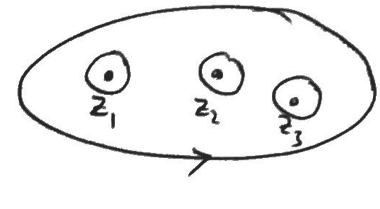
So, $\displaystyle\operatorname*{Res}_{z=0} f(z) = -\frac{1}{3!} = -\frac{1}{6}$.

Hence, $\displaystyle\int_{|z|=1} f(z)\, dz = -2\pi i \cdot \frac{1}{6} = -\frac{\pi i}{3}$.

Residue Theorem :-

Let, C be a simple closed contour, which is positively oriented. If f is analytic inside and on C except for a finite number of singular points z_K $(K=1, 2, \ldots, n)$ inside C, then

$$\int_C f(z)\, dz = 2\pi i \sum_{K=1}^{n} \operatorname*{Res}_{z=z_K} f(z).$$



Calculation of Residues :-

Rule for by-passing Laurent Series Calculation :

An isolated singular point z_0 of f is a pole of order m iff

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}, \text{ where}$$

$\phi(z)$ is analytic and $\phi(z_0) \neq 0$. and

$$\operatorname*{Res}_{z=z_0} f(z) = \phi(z_0), \quad m=1$$

or $$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}, \quad m \geq 2.$$

Suppose $m=4$:

$$f(z) = \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \frac{b_3}{(z-z_0)^3} + \frac{b_4}{(z-z_0)^4} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$= \frac{\sum_{i=1}^{4} b_i (z-z_0)^{4-i} + \sum_{n=0}^{\infty} a_n (z-z_0)^{n+4}}{(z-z_0)^4} \equiv \frac{\phi(z)}{(z-z_0)^4}.$$

$\phi(z_0) = b_4 \neq 0$.

& $\phi(z) = b_1 (z-z_0)^3 + \cdots$

$\therefore \phi^{(3)}(z) = 3.2.1.\, b_1 + \cdots$

$$\Rightarrow b_1 = \frac{\phi^{(3)}(z_0)}{3!} = \operatorname*{Res}_{z=z_0} f(z)$$

EXM. $f(z) = \dfrac{z+1}{z^2+9}$

$z = 3i$ is an isolated singular pt.

$$f(z) = \frac{\phi(z)}{(z-3i)}, \quad \phi(z) = \frac{z+1}{(z+3i)}.$$

$\phi(3i) \neq 0$ & ϕ is analytic at $z = 3i$.

$$\operatorname*{Res}_{z=3i} f(z) = \phi(3i) = \frac{3-i}{6}.$$

EXM. $f(z) = \dfrac{z^3+2z}{(z-i)^3} = \dfrac{\phi(z)}{(z-i)^3}.$

$\phi(i) = -i+2i = i \neq 0$, ϕ is analytic at i.

$z = i$ is a pole of f of order 3.

$$\operatorname*{Res}_{z=i} f(z) = \frac{\phi^{(2)}(i)}{2!} = \frac{6i}{2} = 3i.$$

EXM. $f(z) = z\sin\frac{1}{z}$

$z = 0$ is an isolated essential singular point.

$$f(z) = z\sin\frac{1}{z} = z\sum_{n=0}^{\infty} \frac{1}{z^{2n+1}(2n+1)!} \qquad 0 < |z| < \infty.$$

$$= 1 - \frac{1}{3!z^2} + \cdots$$

$$\therefore \operatorname*{Res}_{z=0} f(z) = 0.$$

$$f(z) = z\cos\left(\frac{1}{z}\right) = z\left(1 - \frac{1}{2!z^2} + \frac{1}{4!z^4} - \cdots\right)$$

$$= z - \frac{1}{2!z} + \frac{1}{4!z^3} - \cdots$$

$$\therefore \operatorname*{Res}_{z=0} f(z) = -\frac{1}{2!} = -\frac{1}{2}.$$

Exm. $\int_C \frac{5z^2}{(z-1)^2(z+3)}\,dz$; C is any simple closed curve

with i) $1, -3$ inside C, ii) $1, -3$ are outside C.

If $1, -3$ are outside C, $f(z) = \frac{5z^2}{(z-1)^2(z+3)}$ is analytic inside and on C.

ii) So, by CIT, $\int_C f(z)\,dz = 0$.

i) $\operatorname*{Res}_{z=1} f(z) = \frac{d}{dz}\left(\frac{5z^2}{z+3}\right)\Big|_{z=1}$

$= \frac{(z+3)\,10z - 5z^2}{(z+3)^2}\Big|_{z=1} = \frac{5z^2+30z}{(z+3)^2} = \frac{35}{16}$.

$\operatorname*{Res}_{z=-3} f(z) = \frac{5z^2}{(z-1)^2}\Big|_{z=-3} = \frac{45}{16}$.

$\therefore \int_C f(z)\,dz \underset{\text{Residue thm}}{=} 2\pi i\left(\frac{35}{16} + \frac{45}{16}\right) = 5.2\pi i = \underline{\underline{10 i\pi}}$.

Exm. $\int_{|z|=3/2} \frac{\tan z}{z^2-1}\,dz$

$\tan z$ is not analytic at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm \ldots$, but they are outside $|z| = \frac{3}{2}$.

$f(z) = \frac{\tan z}{z^2-1}$ has singularity at $z = \pm 1$. And $z = \pm 1$ are poles of f.

$\operatorname*{Res}_{z=1} (f(z)) = \frac{\tan z}{z+1}\Big|_{z=1} = \frac{\tan 1}{2}$

$\operatorname*{Res}_{z=-1} (f(z)) = \frac{\tan z}{z-1}\Big|_{z=-1} = \frac{\tan 1}{2}$.

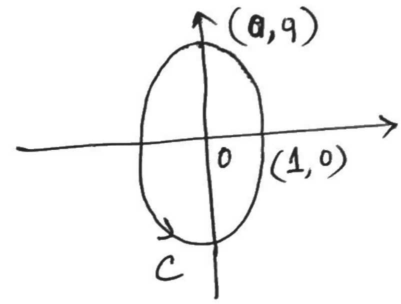
By Residue thm,

$$\int_C f(z)\,dz = 2\pi i\left(\frac{\tan 1}{2} + \frac{\tan 1}{2}\right)$$

$$= \underline{2\pi i \tan 1}$$

<u>EXM</u>

$$\int_C \left(\frac{ze^{\pi z}}{z^4-16} + ze^{\pi/z}\right) dz \qquad C : x^2 + \frac{y^2}{9} = 1.$$



$z = \pm 2i$ are two simple poles

of $\quad f_1(z) = \dfrac{ze^{\pi z}}{z^4-16}$

$$\therefore \quad \operatorname*{Res}_{z=2i}(f_1(z)) = \left.\frac{ze^{\pi z}}{(z^2-4)(z+2i)}\right|_{z=2i}$$

$$= \frac{2i}{-8.4i} = -\frac{1}{16}.$$

$$\operatorname*{Res}_{z=-2i}(f_1(z)) = \left.\frac{ze^{\pi z}}{(z^2-4)(z-2i)}\right|_{z=-2i} = \frac{-2i}{-8.(-4i)} = -\frac{1}{16}.$$

$$\therefore \quad \int_C f_1(z)\,dz = 2\pi i\left(-\frac{1}{16} - \frac{1}{16}\right) = -\frac{2\pi i}{8}.$$

Now, $\quad f_2(z) = ze^{\pi/z} = z\displaystyle\sum_{k=0}^{\infty}\frac{\pi^k}{z^k\,k!}$

$$= z\left(1 + \frac{\pi}{z} + \frac{\pi^2}{2z^2} + \cdots\right)$$

$$= z + \pi + \frac{\pi^2}{2z} + \cdots$$

So, $\quad \operatorname*{Res}_{z=0}(f_2(z)) = \dfrac{\pi^2}{2}.$

Hence, $\quad \displaystyle\int_C f_2(z)\,dz = 2\pi i\,\frac{\pi^2}{2}.$

So, $\quad \displaystyle\int_C f(z)\,dz = 2\pi i\left(\frac{\pi^2}{2} - \frac{1}{8}\right) = \underline{\pi i\left(\pi^2 - \frac{1}{4}\right)}.$

Zeros :–

Let, f is analytic at z_0. So, f has derivaties of all order at z_0. If

$$f(z_0) = 0 = f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0), \text{ but}$$

$$f^{(m)}(z_0) \neq 0 \quad \text{for some } m \in \mathbb{N},$$

then f is said to have a zero of order m at z_0.

$\Rightarrow$ If $m=1$, z_0 is a simple zero of f.

$\Rightarrow$ As f is analytic at z_0,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

If f has zero of order m at z_0,

$$f(z) = \sum_{k=m}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

$$= (z-z_0)^m \sum_{k=0}^{\infty} \frac{f^{(k+m)}(z_0)}{(k+m)!}(z-z_0)^k$$

$$= (z-z_0)^m g(z)$$

with g analytic at z_0 and $g(z_0) \neq 0$.

Exm. $f(z) = z^3 - 8 = (z-2)(z^2+2z+8)$ has zero of order 1 at 2.

Exm. $f(z) = z(e^z - 1)$, $z_0 = 0$.

f is analytic at $z_0 = 0$.

$f(0) = 0 = f'(0)$ $\quad f''(0) \neq 2$

$z_0 = 0$ is a zero of f of order 2.

Thm. Let, p & q are two analytic fns at z_0. If $p(z_0) \neq 0$ & $q(z_0) = 0$ & $q'(z_0) \neq 0$, then z_0 is a simple pole of $f(z) = \frac{p(z)}{q(z)}$ and

$$\operatorname*{Res}_{z=z_0} (f(z)) = \frac{p(z_0)}{q'(z_0)}$$

$\Rightarrow$ $q(z) = (z-z_0) g(z)$, $g(z_0) \neq 0$.

$$\therefore \quad f(z) = \frac{p(z)}{q(z)} = \frac{p(z)/g(z)}{(z-z_0)} = \frac{\phi(z)}{(z-z_0)}, \quad \phi(z_0) \neq 0.$$

f has pole at z_0 of order 1.

So, $$\operatorname*{Res}_{z=z_0} (f(z)) = \phi(z_0) = \frac{p(z_0)}{g(z_0)} = \frac{p(z_0)}{q'(z_0)}$$

Exm. $$f(z) = \cot(z) = \frac{\cos z}{\sin z} = \frac{p(z)}{q(z)}$$

$z = n\pi$ are zeros of $q(z)$.

$p(n\pi) = (-1)^n \neq 0$, $q(n\pi) = 0$, $q'(n\pi) \neq 0$.

So, at $z = n\pi$, f has a simple pole. &

$$\operatorname*{Res}_{z=n\pi} (f(z)) = \frac{p(n\pi)}{q'(n\pi)} = \frac{(-1)^n}{(-1)^n} = 1.$$

Exm $$\int_{|z|=3} \frac{e^z}{(z+1)^n}\, dz$$

$f(z) = \frac{e^z}{(z+1)^n}$. $z = -1$ is a pole of order n.

$$\operatorname*{Res}_{z=-1} (f(z)) = \frac{1}{(n-1)!} \left[\frac{d^{n-1}}{dz^{n-1}} (z+1)^n f(z) \right]_{z=-1} = \frac{e^{-1}}{(n-1)!}$$

$$\therefore \int_C f(z)\, dz = \frac{2\pi i}{e(n-1)!}$$