ODE

BCM ODEs

ODEs:

A. 1st order Equations.

- Existence & Uniqueness Theorems (Picard - Lindelöf & Cauchy - Peano)

B. 2nd order Equations.

C. System of 1st order ODEs

- Stability theory.

D. Power series solutions

- Special functions.

E. Sturm-Liouville Theory.

Books:-

1. An introduction to ODEs,: E. Coddington.
2. Differential Equations.: S.L. Ross
3. Advanced Differential Equations,: M.D. Raisinghania.
4. An introduction to ODEs: R. Agarwal & O'Regan.
5. ODEs, principles & Applications, A Nandakumaran, P. Datti, R. George.
6. ODEs : Tyn Myint-U.

bankimmath.weebly.com/lecnotes.

Few Definitions :-

A. **Lipschitz Continuous :-** A function $f : (a,b)\times D \to \mathbb{R}^n$ is said to be Lipschitz continuous w.r.to y, if $\exists\, \alpha > 0$ such that

$$|f(t,y_1) - f(t,y_2)| \le \alpha |y_1 - y_2| \quad \forall\, t \in (a,b)$$
$$\& \; y_1, y_2 \in D.$$

α is called a Lipschitz constant of f.

EXM :-

$$f(t,y) = e^{-t^2} y^2 \sin t \quad , \quad t \in \mathbb{R}, \quad 0 \le y \le 2.$$

$$|f(t,y_1) - f(t,y_2)| = e^{-t^2} |\sin t|\,|y_1 - y_2|\,|y_1 + y_2|$$
$$\le 4|y_1 - y_2| \qquad \text{Lipschitz Cont.}$$

EXM :-

$$f(t,y) = t\sqrt{y}. \quad , \quad 0 \le t \le 1, \quad 0 \le y \le 1.$$

$$|f(1,y) - f(1,0)| = \sqrt{y} = \frac{1}{\sqrt{y}}|y - 0|$$

and $\frac{1}{\sqrt{y}} \to \infty$ as $y \to 0+$. So, f is not Lipschitz on D, but is cont. on D.

B. **Sufficient Condn :-**

Let, $f : (a,b)\times D \to \mathbb{R}^n$ be a C^1 fn, where, D is a convex domain in $\mathbb{R}^n$ with

$$\sup_{\substack{t\in(a,b)\\ y\in D}} \left| \frac{\partial f_i}{\partial y_j}(t,y) \right| = \alpha < \infty$$

for $i, j = 1, 2, \dots, n$. Then, f is Lipschitz continuous on $(a,b)\times D$ w.r.to y, having a Lipschitz constant

$\leq$ some multiple of α.

Exm:- $f(t,y) = t + y^2$, $|t| \leq a$, $|y| \leq b$.

$$\frac{\partial f}{\partial y} = 2y \quad \& \quad \left|\frac{\partial f}{\partial y}\right| = 2|y| \leq 2b.$$

So, f is Lipschitz with constant $2b$.

Exm:- $f(t,y) = t|y|$, $|t| \leq a$, $|y| \leq b$.

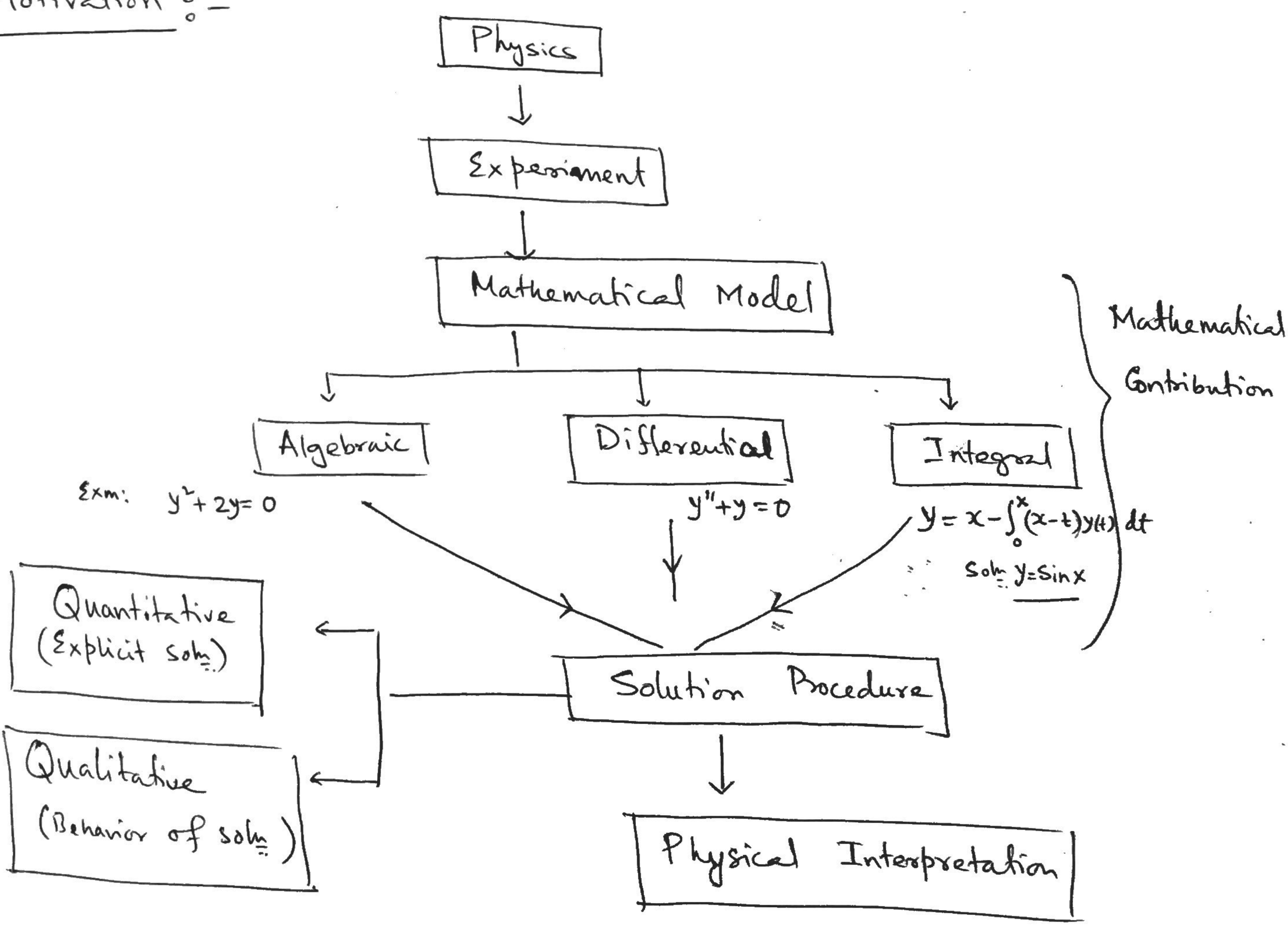
$$|f(t,y_1) - f(t,y_2)| = |t|\,\big||y_1| - |y_2|\big|$$
$$\leq |t|\,|y_1 - y_2| \leq a\,|y_1 - y_2|$$

So, f is Lipschitz cont., but $\frac{\partial f}{\partial y}$ does not exist at $(t,0) \in D$, $t \neq 0$.

So, Continuity $\leq$ Lipscitz $\leq$ Differtiability.

ODEs

Motivation :-



Formulation :- 1. Assume that a spherical rain drop evaporates at a rate proportional to its surface area. Find the radius of the rain drop as a function of t.

Soln :-

Let, V denotes the volume and S the surface area of the rain drop. Then,

$$V = \frac{4}{3}\pi r^3 \ , \text{ where, } r(t) \text{ be the radius}$$

at time t. And

$$S = 4\pi r^2$$

Then,

$$\frac{dV}{dt} \propto S$$

$$\Rightarrow \frac{dV}{dt} = -KS \ , \quad -\text{ve sign for decrasing volume.}$$

$$\Rightarrow \frac{dV}{dr} \cdot \frac{dr}{dt} = -KS$$

$$\Rightarrow 4\pi r^2 \frac{dr}{dt} = -K \cdot 4\pi r^2$$

$$\Rightarrow \frac{dr}{dt} = -K$$

$$\therefore \underline{r(t) = -Kt + K_2}.$$

More examples:

2. Newton's Law of cooling
3. Radio active dating
4. Population Growth
5. Newton's law of motion.

Definition :-

A general form of an ODE is

$$F(t, x, x', x'', \dots, x^{(n)}) = 0 \quad \dots (1)$$

where $t \mapsto x(t)$ is an unknown function of indep variable t.

Standard Form :-

$$x^{(n)} = G(t, x, x', \dots, x^{(n-1)}) . \quad \dots (11)$$

Nature of the Equations :-

I) The highest order derivative of unknown fn x in the relation (1) is called the order of the ODE.

II) The degree of an ODE is the degree of the highest derivative, after the ODE has been made free from radicals and fractions in derivatives.

III) The ODE is linear if G is linear in x and ~~to~~ its derivatives. If it is not linear, it is called non-linear.

An nth-order linear ODE has the form:

$$P_0(t)\, x^{(n)} + P_1(t)\, x^{(n-1)} + \dots + P_n(t)\, x = r(t) .$$

IV) If $r(t) = 0$, it is called homogeneous, else it is non-homogeneous ODE.

EXM :-

$$(x + t x'^2)^{4/3} = t x''$$

$$\Rightarrow (x + t x'^2)^4 = t^3 x''^3$$

So, the ODE is of order 2, degree 3.

Solution of an ODE :-

A function $\phi : I \to \mathbb{R}$ is said to be a solution of

$$F(t, x, x', \ldots, x^{(n)}) = 0 \quad \text{if}$$

i) $\phi \in C^n(I)$,

ii) $F(t, \phi(t), \phi'(t), \ldots, \phi^{(n)}(t)) = 0 \quad \forall t \in I$.

$\Rightarrow$ ϕ is the above definition is called an explicit soln.

$\Rightarrow$ A relation $\psi(t, x) = 0$ is called an implicit soln of (1) if it is possible to fined one or more fns $x = \phi(t)$ from the relation, which is an explicit soln of (1).

$\Rightarrow$ 1. The eqn. $(y')^2 + y^2 + 1 = 0$ has no solution.

2. $x' = \begin{cases} 1, & t \geq 0 \\ -1 & \text{if } t < 0 \end{cases}$ does not have a soln (By Darboux theorem).

3. $x' = tx$, $x(0) = x_0 \Rightarrow X(t) = x_0 e^{t^2/2}$: unique soln.

4. $x' = x^2$, $x(0) = x_0 \Rightarrow x(t) = \dfrac{x_0}{1 - tx_0}$, $t \in (-\infty, 1/x_0)$ with $x_0 > 0$

and $t \in (\frac{1}{x_0}, \infty)$, for $x_0 < 0$

The intervals are coming from the continuity of $x(t)$ in I, i.e. $x(t)$ has the same sign as $x(0)$ in small nbd.

Let, $x_0 = 0$. Then, $x(t) \equiv 0$ is a soln $\forall t \in \mathbb{R}$.

To prove that this is the only soln.

If possible, let us consider another soln $x_1(t)$ such that

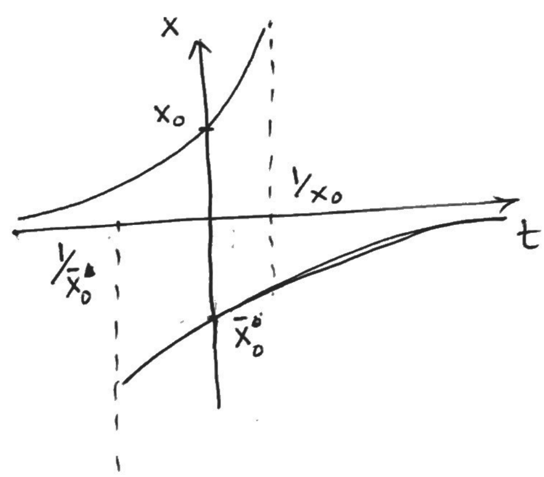
$$x_1(t_1) = x_1^* \neq 0.$$

So, $x = x_1(t)$ is a soln of $x' = x^2$, $x(t_1) = x_1^* \neq 0$.

Then, $x_1(t)$ can never be zero. This contradicts

$$x_1(0) = 0.$$

So, for $x_0 \in \mathbb{R}$, $x' = x^2$, $x(0) = x_0$ has a unique soln, but for different x_0, the solution is defined on different intervals.



5. $x' = |x|^{1/2}$, $x(0) = 0$. ..(*)

$x(t) \equiv 0$ is a soln. of the IVP.

Also, for any $c > 0$, $x(t) = \dfrac{(t-c)^2}{4}$ is a soln for $t \in [c, \infty)$.

Hence,
$$x(t) = \begin{cases} \dfrac{(t-c)^2}{4}, & t \geq c \\ 0, & t < c. \end{cases}$$

is a soln of (*). For $c < 0$,

$$x(t) = \begin{cases} -\dfrac{(t-c)^2}{4}, & t \leq c \\ 0, & t > c. \end{cases}$$

is a soln of (*) on $\mathbb{R}$. So, (*) has infinite solutions on $\mathbb{R}$.

6. $x' = \text{sgn}(t) = \begin{cases} 1, & t>0 \\ 0, & t=0 \\ -1 & t<0 \end{cases}$, $x(0)=0$.

This IVP has no soln. Note $x(t)=|t|$ would satisfy the eqn, but the function $x \notin C'(I)$, any interval I containing $t=0$.

<u>Initial condn :-</u> (And Boundary Condn)

In addition to the diff. eqn (1), if one or more condn are specified at one pt, $t=t_0$, the conditions are called the initial conditions.

If the conditions are specified at more than one pt, the condn are called boundary condn.

<u>IVP :-</u> $y''+y=0$; $y(0)=0$, $y'(0)=1$

Soln : $y(x) = \sin x$

<u>BVP :-</u> $y''+y=0$; $y(0)=0$, $y(\pi/2) = 2$

Soln is : $y(x) = 2\sin x$

<u>Well-Posed Problems :</u> (Hadamard)

An IVP or BVP is called wellposed if

i) The problem has at least one solution (Existence)

ii) The problem has only one solution (Uniqueness)

iii) Continuous dependence on the initial data. (Stability)

<u>PDE</u>

$$\begin{cases} u_{xx}+u_{yy}=0 \\ u(x,0)=0, \quad u_y(x,0)=0 \end{cases} \Rightarrow u(x,y)=0 \text{ is a soln.}$$

$$\begin{cases} u_{xx}+u_{yy}=0 \\ u(x,0)=0, \quad u_y(x,0)=e^{-\sqrt{n}}\sin(nx) \end{cases} \to 0 \text{ as } n\to\infty$$

* <u>ODE</u>

but, $u(x,y) = \frac{1}{n} e^{-\sqrt{n}} \sin(nx) \sinh(ny) \to \infty$ as $y \to \infty$.

Geometric Interpretation of 1^{st} order ODE :-

$$\frac{dy}{dx} = f(x,y) \quad \ldots \quad (*)$$

1. Line Element :- A Line element associated to a pt. $(x,y) \in D \subseteq \mathbb{R}^2$ is a line passing through the pt (x,y) with slope p. A line element is denoted by (x,y,p).

2. Direction field :- A direction field associated to the ODE $(*)$ is collection of all lines elements in the domain D where $p = f(x,y)$. i.e. the set

$$\{ (x,y, f(x,y)) \mid (x,y) \in D \}$$

3. Solving an ODE can be interpreted as finding curves in D that fit the direction field. A solution ϕ of $(*)$ passing through a pt $(x_0, y_0) \in D$ $(\phi(x_0) = y_0)$ must satisfy $\phi'(x_0) = f(x_0, y_0)$.

* $y' = y$, $y(0) = 0$ $\Rightarrow$ $y \equiv 0$ is the solution.

$y' = y$, $y(0) = \epsilon$ $\Rightarrow$ $y(x) = \epsilon e^x$ is the soln.

Now, $\epsilon \neq 0 \Rightarrow |y(x)| \to \infty$ as $x \to \infty$.

Therefore, $y(x) = 0$ is unstable.