System of Equations

General form of Linear system is:

$$x_1'(t) = a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}(t)x_n(t) + g_1(t)$$

$$x_2'(t) = a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}(t)x_n(t) + g_2(t)$$

$$\vdots$$

$$x_n'(t) = a_{n1}(t)x_1(t) + \cdots + a_{nn}x_n(t) + g_n(t)$$

where, x_i's are unknown functions and t is the indep. variable, with $[x_1(0), \ldots x_n(0)]^T = [x_1^0, \ldots, x_n^0]^T$

In matrix form,

$$\bar{X}'(t) = A(t)\bar{X}(t) + \bar{g}(t) \quad \cdots (1)$$

where, $\bar{X}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$, $A(t) = (a_{ij}(t))_{n\times n}$

& $\bar{g}(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix}$ and $\bar{X}(0) = \bar{X}^0$.

⇒ $A(t)$ is called the co-efficient matrix.

⇒ If $\bar{g}(t) = 0 \ \forall t$, then the system is called homogeneous, else non-homogeneous.

⇒ The equation (1) is of the form:

$$\bar{X}'(t) = F(t, \bar{X}(t))$$
$$\& \ \bar{X}(0) = \bar{X}^0.$$

<u>Defn</u>:- If A depends on t, (1) is called a non-autonomous system, else it is an autonomous system.

Theorem 24 (Existence & Uniqueness) :-

If the functions $A(t)$ and $\bar{g}(t)$ are continuous on an open interval $I \subseteq \mathbb{R}$, and $t_0 \in I$, then $\exists$ unique function $\bar{x}(t)$, defined on $I_0 \subset I$ with $t_0 \in I_0$, solution of the IVP:

$$\bar{x}'(t) = A(t)\,\bar{x}(t) + \bar{g}(t), \quad \bar{x}(t_0) = \bar{b} \quad \ldots \text{(II)}$$

Theorem 25 (Superposition Principle)

If $\bar{x}_1(t), \bar{x}_2(t), \ldots, \bar{x}_m(t)$ are solutions of the homogeneous problem

$$\bar{x}'(t) = A(t)\,\bar{x}(t), \quad \text{then}$$

$$\tilde{x}(t) = \sum_{k=1}^{m} c_k \bar{x}_k(t) \text{ is also a solution for any}$$

constants c_k.

Defn :- Let, $\bar{x}_1(t), \ldots, \bar{x}_n(t)$ be n solutions of (II). Then the matrix

$$\bar{\bar{X}}(t) = (\bar{x}_1(t), \ldots \bar{x}_n(t)) \text{ is called a solution}$$

matrix of (II).

If $\bar{x}_i$'s $i = 1(1)n$ are L.I, then $\bar{\bar{X}}(t)$ is called the fundamental matrix of (II)

Theorem 26 Let, $\bar{x}_i(t)$ be n L.I. solns of $\bar{x}'(t) = A(t)\,\bar{x}(t)$. Then for every soln Φ, $\exists$ a unique linear combination of $\bar{x}_i$, for $i = 1, 2, \ldots n$.

Proof :- Let, Φ be any solution of the system

$$\bar{x}'(t) = A(t)\,\bar{x}(t) \text{ with } \bar{x}(t_0) = \bar{b}.$$

Let, $\bar{x}_i(t_0) = b_i$. Since b_i are L.I, $\exists$ c_k s.t.

$$\bar{b} = \sum_{k=1}^{n} c_k b_k$$

Hence the function

$\sum C_k \bar{X}_k$ is a solution of the system.

with the value at t_0 is $\bar{b}$.

So, $\Phi = \sum C_k \bar{X}_k$, by uniqueness theorem.

Exm :-

$\bar{X}' = A\bar{X}$, where $A = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}$.

Verify $\bar{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}$ and $\bar{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t}$ are solutions to the system.

Consider $c_1 \bar{x}_1 + c_2 \bar{x}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$$\Rightarrow [\bar{x}_1 \ \bar{x}_2] \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} e^{-2t} & -e^{+4t} \\ e^{-2t} & e^{4t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \bar{\bar{X}}(t)\, \bar{c} = \bar{0}$$

Now $\det(\bar{\bar{X}}(t)) = \begin{vmatrix} e^{-2t} & -e^{+4t} \\ e^{-2t} & e^{4t} \end{vmatrix} = 2e^{2t} \neq 0$.

i.e. $\bar{\bar{X}}(t)$ is invertible for $t \in \mathbb{R}$.

hence, $\bar{c} = \bar{0}$. $\Rightarrow \{\bar{x}_1, \bar{x}_2\}$ is L.I

Hence it is a fundamental set.

Defn :- The Wronskian of the set $\bar{\bar{X}}(t) = \{\bar{x}_1(t), \dots, \bar{x}_n(t)\}$ is the function $W(t) = \det(\bar{\bar{X}}(t))$.

Theorem 27 If the Wronskian $W(t)$ of n-vector valued functions $\bar{x}_1(t), \dots, \bar{x}_n(t)$ is non-zero for at least one pt in I, then these fns are L.I in I.

Proof :- $\bar{\bar{X}}(t)\,\bar{c} = \bar{0}$ has unique soln $\bar{c} = \bar{0} \Leftrightarrow W(t) \neq 0 \ \forall t \in I$.

Theorem 28 (Abel)

The Wronskian $W(t) = \det(\bar{\bar{X}}(t))$ of a solution matrix $\bar{\bar{X}}(t)$ of the linear system $\bar{X}'(t) = A(t)\,\bar{X}(t)$ satisfies the diff. equation

$$W'(t) = \text{trace}(A(t))\, W(t).$$

Hence W is given by $W(t) = W(t_0)\, e^{\alpha(t)}$,

$$\alpha(t) = \int_{t_0}^{t} \text{trace}(A(\tau))\, d\tau, \quad t_0 \in I.$$

Proof:- Let, $\bar{\bar{X}} = [\bar{X}_1, \bar{X}_2 \ldots \bar{X}_n]$ and x_{ij} be the components of $\bar{X}_j$. Then, for $\bar{X}'(t) = A(t)\,\bar{X}(t)$ we get,

$$x_{ij}'(t) = \sum_{k=1}^{n} a_{ik} x_{kj} \qquad i, j = 1, 2, \ldots, n.$$

Now,

$$W'(t) = \sum_{i=1}^{n} \begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{i1}' & & x_{in}' \\ \vdots & & \vdots \\ x_{n1} & & x_{nn} \end{vmatrix}$$

$$= \sum_{i=1}^{n} \begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \\ \sum a_{ik} x_{k1} & \cdots & \sum a_{ik} x_{kn} \\ \vdots & & \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}$$

Now multiply the 1st row by $a_{i1}(t)$, 2nd row by $a_{i2}(t)$ and so on, except ith row and subtract their sum from the ith row, to get

$$W'(t) = \sum_{i=1}^{n} a_{ii}(t)\, W(t) = \text{trace}(A(t))\, W(t).$$

This is a linear ODE of single fn $W : \mathbb{R} \to \mathbb{R}$.

So, integrating we get

$$W(t) = W(t_0)\, e^{\alpha(t)}, \quad \alpha(t) = \int_{t_0}^{t} \text{trace}(A(\tau))\, d\tau.$$

Remark :-

I) For autonomous system, we have

$$W(t) = W(t_0)\, e^{\operatorname{trace}(A)\,(t-t_0)}$$

II) $W(t) = 0 \quad \forall t \iff W(t_0) = 0$ for some $t_0 \in I$.

III) $\det(\tilde{X}(t)) = 0 \not\Rightarrow$ column vectors are L.D.

$$\tilde{X}(t) = \begin{pmatrix} t & t^2 \\ 0 & 0 \end{pmatrix}$$

For validity of thm 28, $\tilde{X}(t)$ must be a solution matrix.

Non-homogeneous Linear System :-

Consider the non-homogeneous linear system:

$$\bar{X}'(t) = A(t)\,\bar{X}(t) + \bar{G}(t), \quad t \in I. \quad \cdots (\text{III})$$

Thm 29 (Variation of parameters)

If $\bar{\bar{X}}(t)$ is a fundamental matrix of $\bar{X}'(t) = A(t)\,\bar{X}(t)$, then for any $t_0 \in I$,

$$\Phi(t) = \bar{\bar{X}}\,\bar{c} + \bar{\bar{X}}(t) \int_{t_0}^{t} \bar{\bar{X}}^{-1}(s)\,\bar{G}(s)\,ds$$

is a general soln of (III), where $\bar{c}$ is an arbitrary column vector.

Proof :- Let, the gen. soln of (III) is:

$$\Phi(t) = \bar{\bar{X}}(t)\,\bar{c} + \Phi^*(t)$$

where, $\Phi^*(t)$ is a particular soln of (III).

Let, a particular soln of (III) be of the form:

$$\Phi^*(t) = \bar{\bar{X}}(t)\,\bar{V}(t).$$

Now, $(\Phi^*)'(t) = \bar{\bar{X}}'(t)\,\bar{V}(t) + \bar{\bar{X}}(t)\,\bar{V}'(t)$

$$= A(t)\,\bar{\bar{X}}(t)\,\bar{V}(t) + \bar{\bar{X}}(t)\,\bar{V}'(t)$$

This should be equal to : $A(t)\,\Phi^*(t) + \bar{G}(t)$

This gives: $\bar{\bar{X}}(t)\,\bar{V}'(t) = \bar{G}(t)$

$$\Rightarrow \quad \bar{V}'(t) = \bar{\bar{X}}^{-1}\,\bar{G}(t)$$

Integrating between t_0 to t, we get

$$\bar{V}(t) = \int_{t_0}^{t} \bar{\bar{X}}^{-1}(s)\,\bar{G}(s)\,ds + \bar{K}.$$

So, $\Phi^*(t) = \bar{\bar{X}}(t)\,\bar{V}(t)$

$$= \bar{\bar{X}}(t)\int_{t_0}^{t} \bar{\bar{X}}^{-1}(s)\,\bar{G}(s)\,ds + \bar{\bar{X}}(t)\,\bar{K}.$$

Hence the gen. sol$^{\underline{n}}$ is given by:

$$\Phi(t) = \bar{\bar{X}}(t)\,\bar{C} + \bar{\bar{X}}(t)\int_{t_0}^{t} \bar{\bar{X}}^{-1}(s)\,\bar{G}(s)\,ds.$$

NOTE :-

The solution of the non-homogeneous system (III) satisfying $\Phi(t_0) = \bar{b}$. is given by :

$$\Phi(t) = \bar{\bar{X}}(t)\,\bar{\bar{X}}^{-1}(t_0)\,\bar{b} + \bar{\bar{X}}(t)\int_{t_0}^{t} \bar{\bar{X}}^{-1}(s)\,\bar{G}(s)\,ds.$$

Exponential Matrix :-

Defn (Matrix Norm) Let M_n denotes the set of all $n\times n$ matrices. The fn $\|\cdot\| : M_n \to \mathbb{R}$ is called the matrix norm if $\forall\, A, B \in M_n$

i) $\|A\| \geq 0$

ii) $\|A\| = 0 \Leftrightarrow A = 0$

iii) $\|\alpha A\| = |\alpha|\,\|A\|$, $\alpha \in \mathbb{R}$

iv) $\|A+B\| \leq \|A\| + \|B\|$

Exm: $\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$ (column sum norm)

$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$ (row sum norm)

$\|A\|_F = \left(\sum_i \sum_j |a_{ij}|^2\right)^{1/2}$ (Frobenius norm)

$\|A\|_2 = \sqrt{\lambda_{max}(A^* A)}$ $\quad A^* = (\bar{A})^T$

$\|A\| = \sup_{\|x\|=1} \|Ax\|$ (Induced norm).

Defn: A seqn of matrices $\{A_k\}_k$, where $A_k \in M_n$, is said to converge to $A \in M_n$ as $k \to \infty$ if for each $\epsilon > 0$, $\exists$ an integer $N > 0$ s.t.

$$\|A_k - A\| < \epsilon \quad \forall\, k \geq N.$$

Lemma 12: Let, $A, B \in M_n$ and $\bar{x} \in \mathbb{R}^n$. Then for induced norm,

1) $\|A\bar{x}\| \leq \|A\| \, \|\bar{x}\|$

ii) $\|AB\| \leq \|A\| \, \|B\|$

iii) $\|A^k\| \leq \|A\|^k$, $k = 0, 1, 2, \ldots$

Ans: 1) If $\bar{x} = \bar{0}$, the result is obvious.

Let, $\bar{x} \neq \bar{0}$. Define $\bar{y} = \frac{\bar{x}}{\|\bar{x}\|}$ so that $\|\bar{y}\| = 1$.

Now $\|A\| = \sup_{\|\bar{x}\|=1} \|A\bar{x}\|$

Hence $\|A\| \geq \|A\bar{y}\| = \frac{1}{\|\bar{x}\|} \|A\bar{x}\|$

$\Rightarrow \|A\bar{x}\| \leq \|A\| \, \|\bar{x}\|$

ii) For $\|\bar{x}\| = 1$,

$$\begin{aligned} \|AB\bar{x}\| &= \|A(B\bar{x})\| \\ &\leq \|A\| \, \|B\bar{x}\| \\ &\leq \|A\| \, \|B\| \, \|\bar{x}\| = \|A\| \, \|B\| \end{aligned}$$

So, $\|AB\| = \sup_{\|\bar{x}\|=1} \|AB\bar{x}\| \leq \|A\| \|B\|$.

III). Use Mathematical Induction to show.

Lemma 13

Given $A \in M_n$ & $t_0 > 0$, the series $\sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$ is absolutely & uniformly convergent for all $|t| \leq t_0$.

Defn :- Let, $A \in M_n$. Then for $t \in \mathbb{R}$

$$e^{tA} := \sum_{k=0}^{\infty} \frac{A^k}{k!} t^k \in M_n.$$

Lemma 14. For $A \in M_n$,

i) $\|e^{tA}\| \leq e^{\|A\| |t|}$

ii) if P is an invertible matrix s.t. $A = PBP^{-1}$, $B \in M_n$, then $e^A = Pe^B P^{-1}$

iii) if $P^{-1}AP = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, then $e^{At} = P \operatorname{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t}) P^{-1}$.

iv) if $AB = BA$ for $A, B \in M_n$, then $e^{A+B} = e^A \cdot e^B$.

v) the inverse of the exponential matrix,

$$(e^A)^{-1} = e^{-A}.$$

Proof :-

i)
$$\|e^{tA}\| = \left\| \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \right\|$$
$$\leq \sum_{k=0}^{\infty} \left\| \frac{A^k t^k}{k!} \right\|$$
$$\leq \sum_{k=0}^{\infty} \frac{\|A\|^k |t|^k}{k!} = e^{\|A\| |t|}$$

$$\therefore \|e^{tA}\| \leq e^{|t| \|A\|}$$

ii)

$$e^A = \underset{N\to\infty}{Lt} \sum_{K=0}^{N} \frac{(PBP^{-1})^K}{K!}$$

$$= \underset{N\to\infty}{Lt} \sum_{K=0}^{N} \frac{PB^K P^{-1}}{K!}$$

$$= P\left[\underset{N\to\infty}{Lt} \sum_{K=0}^{N} \frac{B^K}{K!}\right] P^{-1}$$

$$= Pe^B P^{-1}.$$

iii)

$$A = P \operatorname{diag}(\lambda_1, \ldots, \lambda_n) P^{-1}$$

$$= PBP^{-1}, \text{ say.}$$

So, $$e^{At} = Pe^{Bt}P^{-1}$$

$$= P \begin{pmatrix} \sum_{K=0}^{\infty} \frac{\lambda_1^K t^K}{K!} & 0 & \cdots & 0 \\ 0 & \sum \frac{\lambda_2^K t^K}{K!} & & \\ \vdots & & \ddots & \\ 0 & \cdots & & \sum \frac{\lambda_n^K t^K}{K!} \end{pmatrix} P^{-1}$$

$$= P \operatorname{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t}) P^{-1}.$$

iv) Since $AB = BA$, we have

$$(A+B)^K = \sum_{i=0}^{K} \binom{K}{i} A^i B^{K-i}$$

So, $$e^{A+B} = \sum_{K=0}^{\infty} \frac{(A+B)^K}{K!}$$

$$= \sum_{K=0}^{\infty} \sum_{i=0}^{K} \frac{A^i B^{K-i}}{i!(K-i)!}$$

$$= \left(\sum_{J=0}^{\infty} \frac{A^J}{J!}\right)\left(\sum_{m=0}^{\infty} \frac{B^m}{m!}\right)$$

$$= e^A . e^B.$$

v) Since $A.(-A) = (-A).A = -A^2$,

$$e^A . e^{-A} = e^{A-A} = I.$$

By definition $$(e^A)^{-1} = e^{-A}$$

Lemma 15. Let, A be an $n\times n$ matrix, then

$$\frac{d}{dt} e^{At} = A e^{At}.$$

Proof:-

$$\frac{d}{dt} e^{At} = \lim_{h\to 0} \frac{e^{A(t+h)} - e^{At}}{h}$$

$$= \lim_{h\to 0} e^{At} \frac{e^{Ah} - I}{h} \qquad \left[\because (At)(Ah) = (Ah).(At)\right.$$

$$= e^{At} \lim_{h\to 0} \left[A + \frac{A^2 h}{2!} + \cdots\right]$$

$$= e^{At} A.$$

Theorem 30 [Fundamental thm of Linear system]

Let, $A \in M_n$. Then for the system

$\bar{x}' = A\bar{x}$, a fundamental matrix is given by $\bar{\bar{x}} = e^{tA}$. Moreover, for a given $\bar{b} \in \mathbb{R}^n$, the solution of the IVP:

$\bar{x}' = A\bar{x}$, $\bar{x}(t_0) = \bar{b}$ is given by:

$$\bar{x}(t) = e^{(t-t_0)A}\,\bar{b}.$$

Proof:- For $\bar{\bar{x}} = e^{tA}$, $\bar{\bar{x}}' = Ae^{tA} = A\bar{\bar{x}}$, $t \in I$.

So, $\bar{\bar{x}}(t) = e^{tA}$ is a solution matrix.

Moreover, $\bar{\bar{x}}(0) = \mathbb{I}$, hence, $W(t) = \det(\bar{\bar{x}}) = e^{\text{trace}(A).t}$

Thus, $\bar{\bar{x}}(t)$ is a fundamental matrix.

Let, $\bar{x}(t)$ be any solution of the IVP. and let,

$\Phi(t) = e^{-tA}\,\bar{x}(t)$.

Then,

$$\bar{\Phi}'(t) = e^{-tA}\bar{x}'(t) - Ae^{-tA}\bar{x}(t)$$

$$= e^{-tA}\left[\bar{x}'(t) - A\bar{x}(t)\right]$$

$$= \bar{0}$$

Hence $\bar{\Phi}(t) = \bar{c}$, a constant.

Thus, $\bar{\Phi}(t_0) = \bar{c} = e^{-t_0 A}\bar{x}(t_0)$.

Therefore, $\bar{x}(t) = e^{tA}\bar{\Phi}(t)$

$$= e^{(t-t_0)A}\bar{x}(t_0).$$

$$= e^{(t-t_0)A}\bar{b}.$$

NOTE :- For non-homogeneous system with constant co-eff.

$$\bar{x}' = A\bar{x} + \bar{G}(t), \quad \bar{x}(t_0) = \bar{b},$$ we

get the soln by Thm 29,

$$\bar{x}(t) = e^{(t-t_0)A}\bar{b} + \int_{t_0}^{t} e^{(t-\tau)A}\bar{G}(\tau)\,d\tau.$$

Q. How to evaluate e^{tA}, for a given $A \in M_n$?

EXM. 1. $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

2. $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

1. $e^{tA} = \operatorname{diag}(e^t, e^{2t}) = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$

2. $e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \cdots = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} 1 & 6 \\ 0 & 4 \end{pmatrix} + \frac{t^3}{3!}\begin{pmatrix} 1 & 14 \\ 0 & 8 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} e^t & g(t) \\ 0 & e^{2t} \end{pmatrix}, \quad g(t) = 2t + 3t^2 + \frac{14}{3!}t^3 + \cdots$$

Computation of e^{tA} :-

1. Diagonal matrix, $A = \text{diag}(\lambda_1, \dots, \lambda_n)$: $\bar{\bar{X}}(t) = \text{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t})$
2. Diagonalizable Matrices: Let, $P^{-1}AP = D$.

$$\text{Then, } \quad e^{At} = P e^{Dt} P^{-1}.$$

Lemma 16 If the eigen-values λ_1, λ_2 of $A \in M_2$ are real & distinct, then any set of corresponding eigen-vectors $\{\bar{v}_1, \bar{v}_2\}$ forms a basis for $\mathbb{R}^2$.

Also, $P = [\bar{v}_1 \ \bar{v}_2]$ is invertible and $P^{-1}AP = \text{diag}(\lambda_1, \lambda_2)$

Proof :- Consider $c_1 \bar{v}_1 + c_2 \bar{v}_2 = \bar{0}$

Then $c_1 A \bar{v}_1 + c_2 A \bar{v}_2 = \bar{0}$

$\Rightarrow c_1 \lambda_1 \bar{v}_1 + c_2 \lambda_2 \bar{v}_2 = \bar{0}$

Also, $\lambda_1 (c_1 \bar{v}_1 + c_2 \bar{v}_2) = \bar{0}$.

$\Rightarrow c_2 (\lambda_1 - \lambda_2) \bar{v}_2 = \bar{0}$.

Since $\lambda_1 \neq \lambda_2$ & $\bar{v}_2 \neq \bar{0}$, $c_2 = 0$.

Similarly, $c_1 = 0$, Hence $\bar{v}_1, \bar{v}_2$ are L.I.

Therefore $P = [\bar{v}_1 \ \bar{v}_2]$ is invertible.

Now
$$
\begin{aligned}
P \, \text{diag}(\lambda_1, \lambda_2) &= [\lambda_1 \bar{v}_1 \ \ \lambda_2 \bar{v}_2] \\
&= [A \bar{v}_1 \ \ A \bar{v}_2] \\
&= A [\bar{v}_1 \ \ \bar{v}_2] \\
&= AP.
\end{aligned}
$$

So, $P^{-1}AP = \text{diag}(\lambda_1, \lambda_2)$.

Exm. $\bar{X}' = A\bar{X}$, $A = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$.

$$\bar{x}(t) = e^{tA}\,\bar{c} \quad \text{where } \bar{c} = \bar{x}(0).$$

$$= \operatorname{diag}(e^{-t}, e^{2t})\,\bar{c}.$$

So, $x_1(t) = c_1 e^{-t}$, $x_2(t) = c_2 e^{2t}$.

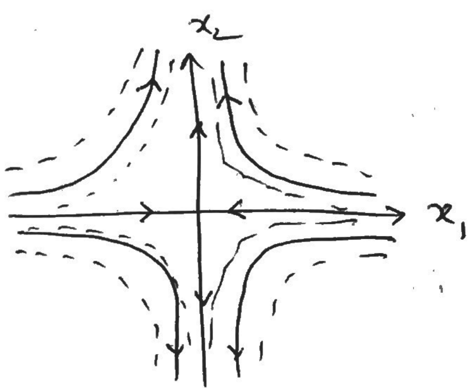
Phase Portrait :-

The phase portrait of a 2×2 system of linear eqn with $\bar{X} : I \to \mathbb{R}^2$ is the set of all solutions of the system in the phase plane $\mathbb{R}^2$.

We have $x_1^2 x_2 = K$.

(i) $c_1 > 0, c_2 > 0 \Rightarrow$ the curve lie on 1st co-ordinate

As $t \uparrow$, $x_1 \downarrow$ & $x_2 \uparrow$



(ii) if $\bar{c} = (c_1, 0)$, $x_1 = c_1 e^{-t}$, $x_2 = 0 \ \forall t$

So, as $t \uparrow$, $x_1 \downarrow$.

Exm. $\bar{X}' = A\bar{X}$, $A = \begin{pmatrix} -1 & -3 \\ 0 & 2 \end{pmatrix}$

$\lambda_1 = -1$, $\lambda_2 = 2$ and the corresponding eigen-vectors are $\bar{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\bar{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Consider $P = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ so that $P^{-1}AP = \operatorname{diag}(\lambda_1, \lambda_2)$

So, $\bar{x}(t) = P \operatorname{diag}(e^{-t}, e^{2t})\, P^{-1}\,\bar{c}. = c_1\bar{v}_1 e^{-t} + c_2 \bar{v}_2 e^{2t}$

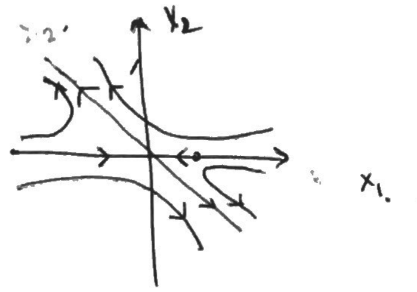
$$\therefore \quad x_1(t) = c_1 e^{-t} + c_2(e^{-t} - e^{2t})$$

$$x_2(t) = c_2 e^{2t}$$

With the transformation $\bar{y} = P^{-1}\bar{x}$, the system reduces to $\bar{y}' = D\bar{y}$

$x_1 = y_1 + y_2$

$x_2 = y_2$



Theorem : Every 2×2 System $\bar{X}' = A\bar{X}$ with $A \in M_2$ is linearly equivalent to one of the system $\bar{Y}' = B_i\bar{Y}$

where
$$B_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad B_2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad B_3 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

NOTE :-

(I) Two diff eig. values or repeated eigen-value with two L.I e. vectors. $\Rightarrow$ $P = [\bar{v}_1 \ \bar{v}_2]$ with $P^{-1}AP = B_1$

(II) Two repeated eigen-values with one eigen-vector.
$\Rightarrow$ $P = [\bar{v}_1 \ \bar{v}_2]$ of generalized eigen-vectors with $P^{-1}AP = B_2$

(III) Two complex eigen-values $a \pm ib$. with e-vectors $v_1 \pm iv_2$
$\Rightarrow$ $P = [\bar{v}_2 \ \bar{v}_1]$ with $P^{-1}AP = B_3$.

Defn : [Generalized Eigen Vectors] Let, λ be an eigen-value of $A \in M_n$ with multiplicity $m \le n$. Then, for a fixed $k \in \{1, 2, \dots, n\}$, any non-zero solu of
$$(A - \lambda I)^k \bar{v} = \bar{0}$$
is called a generalized eigen-vector corresponding to λ.

Exm. $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$

$\lambda = 2, 2.$

For eigen-vector, $(A - 2I)\bar{v} = \bar{0}$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad v_2 = -v_1$$

So, $\bar{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigen-vector.

For generalized eigen-vectors,

$$(A - 2I)^2 \bar{u} = \bar{0}.$$

Let, $(A - 2I)\bar{u} = \bar{v}$., then $(A - 2I)\bar{v} = \bar{0}$.

Then, $\bar{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so that

$$(A - 2I)\bar{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \Rightarrow u_1 + u_2 = 1$$

$$\Rightarrow u_2 = 1 - u_1$$

So, $\bar{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + u_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $u_1 \in \mathbb{R}$. are gen. eigen-vectors.

Lemma 17. If $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, then

$$e^{tA} = e^{at} \begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix}.$$

Proof:- If $\lambda = a + ib$, then by Mathematical Induction, we can prove that,

$$A^k = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^k = \begin{pmatrix} Re(\lambda^k) & -Im(\lambda^k) \\ Im(\lambda^k) & Re(\lambda^k) \end{pmatrix}$$

So, $$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$

$$= \sum_{k=0}^{\infty} \begin{pmatrix} Re\left(\frac{\lambda^k t^k}{k!}\right) & -Im\left(\frac{\lambda^k t^k}{k!}\right) \\ Im\left(\frac{\lambda^k t^k}{k!}\right) & Re\left(\frac{\lambda^k t^k}{k!}\right) \end{pmatrix}$$

$$= \begin{pmatrix} Re(e^{\lambda t}) & -Im(e^{\lambda t}) \\ Im(e^{\lambda t}) & Re(e^{\lambda t}) \end{pmatrix}$$

$$\therefore e^{tA} = \begin{pmatrix} e^{at}\cos bt & -e^{at}\sin bt \\ e^{at}\sin bt & e^{at}\cos bt \end{pmatrix}$$

$$= e^{at}\begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix}$$

Lemma 18. If $A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$, then

$$e^{tA} = e^{\lambda t}\begin{pmatrix} 1 & bt \\ 0 & 1 \end{pmatrix}$$

Proof :- $A = \lambda I + B$ where $B = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$.

Since λI and B commutes, we get

$$e^{tA} = e^{\lambda t I + Bt} = e^{\lambda t I} \cdot e^{tB}$$

$$= e^{\lambda t I} \cdot (I + tB) \quad [\because B^2 = 0]$$

$$= e^{\lambda t} \cdot I\,(I + tB)$$

$$= e^{\lambda t}\begin{pmatrix} 1 & bt \\ 0 & 1 \end{pmatrix}.$$

Exm. $\bar{X}' = A\bar{X}$, $A = \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}$.

$\lambda_1 = 1+2i$, $\lambda_2 = 1-2i$ and corresponding e.vectors are

$\omega_1 = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$, $\omega_2 = \begin{pmatrix} 1 \\ -2i \end{pmatrix}$ so that, $\omega_1 = v_1 + i v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

Let, $P = [v_2 \;\; v_1] = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$, then $P^{-1}AP = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = B$

So, $e^{tA} = Pe^{tB}P^{-1} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} e^{t}\begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$

The soln is: $\bar{X}(t) = e^{tA} \cdot \bar{c}$.

$$\therefore \; x_1(t) = e^t\left(c_1 \cos 2t + \frac{c_2}{2}\sin 2t\right)$$

$$x_2(t) = e^t\left(-2c_1 \sin 2t + c_2 \cos 2t\right)$$

If we take the transformation, $\bar{Y} = P^{-1}\bar{X}$

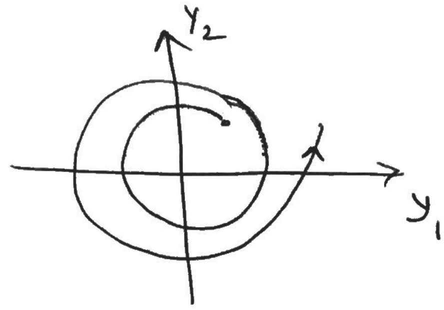
then $\bar{Y}' = P^{-1}\bar{X}'$

$$= P^{-1}AP\,\bar{Y} = B\bar{Y}.$$

The gen. soln of $\bar{Y}' = B\bar{Y}$ is:

$$\bar{Y}(t) = e^t\begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix}\bar{C}.$$

The phase-portrait is:

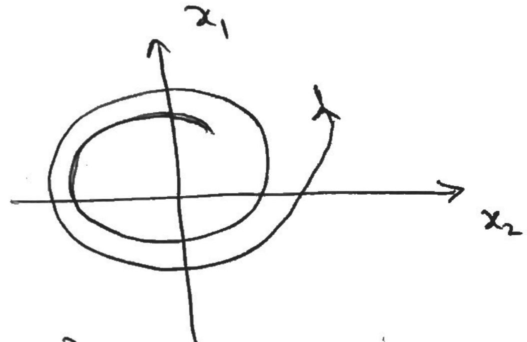


$\bar{Y}$ plane

$\bar{X} = P\bar{Y}$. gives $\quad x_1 = y_2$

$\quad x_2 = 2y_1$

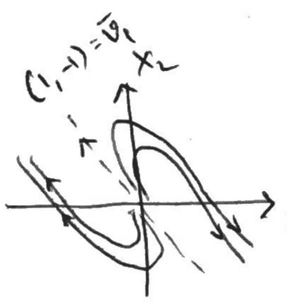
So, the phase-portrait is:



Exm. $\quad \bar{X}' = A\bar{X}, \quad A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$

$\lambda_1 = \lambda_2 = 2$. Then $\exists\, P$ s.t. $P^{-1}AP = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

$$= 2I + B$$



Now, $\quad A = 2I + PBP^{-1}$

$= 2I + N \quad \left[\because P^{-1}NP = P^{-1}(A-2I)P = B\right]$

$$\therefore e^{tA} = e^{2t}(I + Nt) = e^{2t}\begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix}.$$

The soln is: $\quad \bar{X}(t) = e^{tA}\,\bar{C} = e^{2t}\begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$

Lemma 19. Let, $A \in M_2$ with two repeated real eigen-values λ. Let, $A = \lambda I + N$, then

$$e^{tA} = e^{\lambda t}(I + Nt)$$

Proof :- If $P = [\bar{v}_1, \bar{v}_2]$ be the matrix with generalized eigen-vectors $\bar{v}_1$ and $\bar{v}_2$, then

$$P^{-1}AP = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \lambda I + B \quad \text{with}$$

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

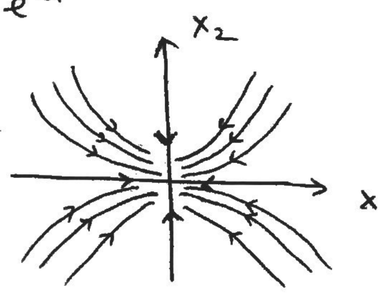
Let, $N = A - \lambda I$, then

$$A = \lambda I + PBP^{-1} \qquad \left[\because P^{-1}NP = P^{-1}(A - \lambda I)P \right.$$
$$= \lambda I + N. \qquad \left. = (\lambda I + B) - \lambda I = B \right]$$

Now, $$e^{tA} = e^{t(\lambda I + N)}$$
$$= e^{\lambda t I} \cdot e^{tN} \qquad [\because N \text{ and } I \text{ commute}]$$
$$= e^{\lambda t} \cdot (I + tN) \qquad [N \text{ is nilpotent}, N^2 = 0]$$

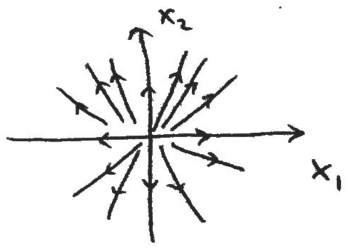
Examples :-

1. $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$ $\qquad \bar{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}$



If $\lambda_1 < \lambda_2 < 0$: the orbits behave as $\bar{v}_2$, since $e^{\lambda_2 t}$ is much bigger than $e^{\lambda_1 t}$ as $t \to \infty$.

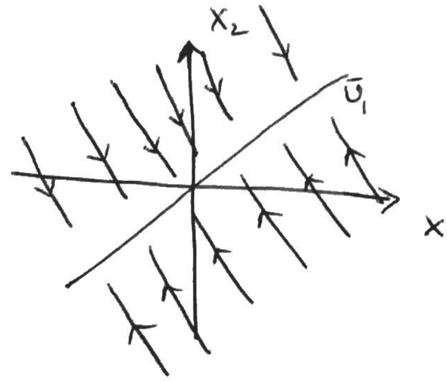
2. $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\qquad \bar{x}(t) = \text{diag}(e^{2t}, e^{2t})\, \bar{c}$.



3. $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$, $\lambda_1 = 0$, $\lambda_2 = -2$

$\bar{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\bar{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\bar{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$$

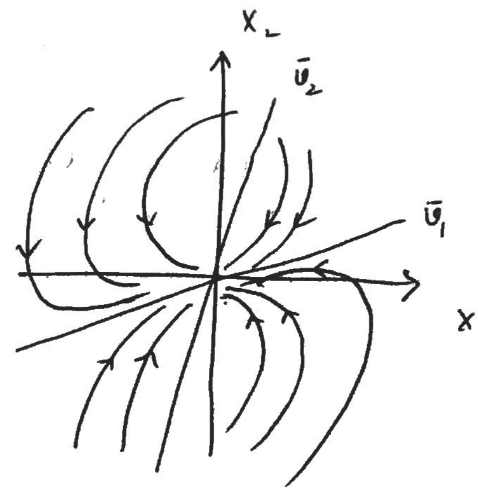


The line of dir. $\bar{v}_1$ is the line of equilibrium points.

4. $A = \begin{pmatrix} 0 & -1 \\ 8 & -6 \end{pmatrix}$ $\lambda_1 = -2$, $\lambda_2 = -4$

$\bar{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\bar{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$$\bar{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-4t}$$



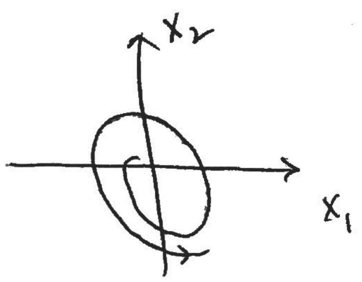
5. $x'' - 6x' + 25x = 0$

$x = x_1$, $x' = x_2$ then the system becomes:

$\bar{x}'(t) = A\bar{x}(t)$, $A = \begin{pmatrix} 0 & 1 \\ -25 & 6 \end{pmatrix}$

$\bar{x}(t) = P e^{3t} \begin{pmatrix} \cos 4t & -\sin 4t \\ \sin 4t & \cos 4t \end{pmatrix} P^{-1} \bar{c}$ with

$P = \begin{pmatrix} -4 & 3 \\ 0 & 25 \end{pmatrix}$.



For system $\bar{y}' = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \bar{y}$

For $z = y_1 + i y_2$ we have

$\dot{z} = \lambda z$, $\lambda = a + ib$

Let, $z = \rho e^{i\theta}$, Then $\dot{\rho} e^{i\theta} + i\dot{\theta}\rho e^{i\theta} = (a+ib)\rho e^{i\theta}$

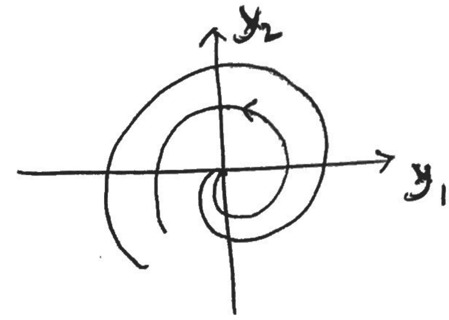
$\Rightarrow \dot{\rho} = a\rho$ and $\dot{\theta} = b$

$\Rightarrow \rho(t) = c_1 e^{at}$, $\theta(t) = bt + c_2$.

$a, b > 0 \Rightarrow \quad \rho \to \infty$ as $t \to \infty$

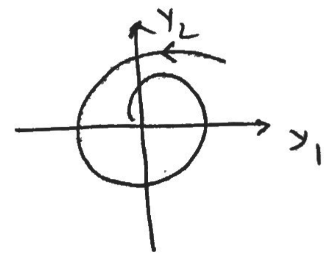
$\rho \to 0$ as $t \to -\infty$

polar angle changes in +ve direction.



$a < 0 : \quad \rho \to 0$ as $t \to \infty$

$\rho \to \infty$ as $t \to -\infty$



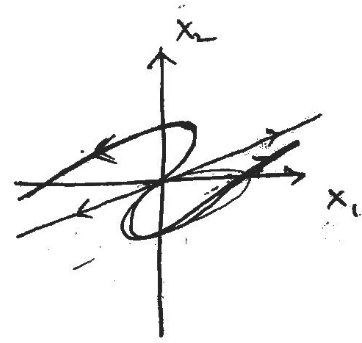
6 $\quad A = \begin{pmatrix} 7 & -4 \\ 1 & 3 \end{pmatrix}, \quad \lambda_1 = \lambda_2 = 5$

$\bar{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\bar{X}(t) = e^{5t} \begin{pmatrix} 1+2t & -4t \\ t & 1-2t \end{pmatrix} \cdot \bar{c}$

$\bar{c} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \quad x_1(t) = (1+2t)\, e^{5t}$

$x_2(t) = t\, e^{5t}$



$\lim_{t \to \infty} \frac{dx_1}{dx_2} = 2$

And $x_1, x_2 \to \infty$ as $t \to \infty$.

Defn :-

A point $\bar{c}$ in $\mathbb{R}^n$ at which $f(\bar{c}) = \bar{0}$, is called an equilibrium or critical point of the autonomous system

$$\bar{x}' = f(\bar{x}) \quad \text{with } f: \mathbb{R}^n \to \mathbb{R}^n.$$

$$\ldots (*)$$

Defn :- (Stability of a system)

Let, $\bar{c}$ be an equilibrium pt. for $(*)$. Then the pt. $\bar{c}$ is said to be:

a) Stable if for any $\epsilon > 0$, $\exists\, \delta > 0$ s.t

$$|\bar{x}(t) - \bar{c}| < \epsilon \quad \text{whenever } |\bar{x}(0) - \bar{c}| < \delta$$
$$\forall t > 0$$

b) Asymptotically stable if $\exists\, \delta > 0$ s.t.

$$\lim_{t \to \infty} |\bar{x}(t) - \bar{c}| = 0 \quad \text{whenever } |\bar{x}(0) - \bar{c}| < \delta.$$

c) Unstable if it is not stable.

A. If the trajectory (soln curve in $\mathbb{R}^n$) approches the critical pt in spiral like manner as $t \to \infty$ or $t \to -\infty$, the critical pt is called a spiral or focus.

B. If all the trajectories moves around the critical pt along a closed path, and never approches to the critical pt as $|t| \to \infty$, then the critical pt is called a center.

C. A critical pt is called a saddle pt if
- four branches of traj., st. lines dividing into four regions
- out of these four st. lines, two approches to the

critical pt as $t \to \infty$ and two approches as $t \to -\infty$.

- in each of these four regions infinite semi-hyperbolic trajectories are there, that do not approches to the critical pt as $|t| \to \infty$, but becomes asymptotic to one of the half-line trajectories.

D. If all the trajectories are st. lines, and approches to the critical pt as $|t| \to \infty$, then it is called a proper node.

E. A critical pt is called an improper node if
- two or four rectilinear traj. divides the $x_1 x_2$ plane into two or four regions
- all the traj. approches to the critical pt as $|t| \to \infty$
- all other traj. are semi-parabolic.

Eigen-values	Type of critical pt	Stability
$\lambda_1 > \lambda_2 > 0$	Improper node	unstable
$\lambda_1 < \lambda_2 < 0$	"	asymp. stable
$\lambda_1 < 0 < \lambda_2$	saddle pt.	unstable
$\lambda_1 = \lambda_2 > 0$	proper or Improper Node	"
$\lambda_1 = \lambda_2 < 0$	"	Asymp. Stable
$\lambda_1, \lambda_2 = a \pm ib$	Spiral	$a > 0$ unstable $a < 0$ asymp. stable
$\lambda_1, \lambda_2 = \pm ib$	Center	Stable.

Problems :-

1. $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$, $A = \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}$

$A = \begin{pmatrix} -1 & 1 \\ 0 & 1/2 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ -6.25 & 0 \end{pmatrix}$

Solve the system $\bar{x}' = A\bar{x}$ and draw the phase-portraits.

2. Let, $\lambda \in \mathbb{R}$ and $\bar{v} \in \mathbb{R}^n$. show that, $\bar{x}(t) = e^{\lambda t}\bar{v}$ is a solution of $\bar{x}' = A\bar{x}$, $A \in M_n$ iff $A\bar{v} = \lambda\bar{v}$.

3. Suppose $A \in M_n$ has a neg. eigen-value. Show that the linear system $\bar{x}' = A\bar{x}$ has at least one non-trivial soln that satisfies $\lim_{t \to \infty} \bar{x}(t) = \bar{0}$.

4. Solve as system $x'' - 3x' + 2x = 0$ and draw its phase-portraits.

5. Let, $A \in M_2$ has real repeated eigen-values λ. If $\bar{v}_2$ is an generalized eigen-vector of A, then prove that

$$A\bar{v}_2 = \alpha\bar{v}_1 + \lambda\bar{v}_2$$ with $\alpha \neq 0$, $\bar{v}_1$ being an eigen-vector corresponding to λ.

6. If the eigen-values λ_1 and λ_2 of A are complex where $\lambda_1, \lambda_2 = a \pm ib$ with eigen-vectors $\omega_1, \omega_2 = \bar{u} \pm i\bar{v}$. Then, prove that the set of vectors $\{\bar{u}, \bar{v}\}$ forms a basis for $\mathbb{R}^2$.

Moreover, $P = [\,\bar{v} \;\; \bar{u}\,]$ is invertible &

$$P^{-1}AP = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$