General Form :-

BCM ODEs

$$F(t, x, x') = 0 , \quad t \in I .$$

It can be written in normal form :

$$x' = f(t, x) .$$

Defn (System of Eqn) :- A system of 1st order eqn takes the form :

$$\frac{dx_1}{dt} = f_1(t, x_1, x_2, \ldots, x_n)$$

$$\frac{dx_2}{dt} = f_2(t, x_1, x_2, \ldots, x_n)$$

$$\vdots$$

$$\frac{dx_n}{dt} = f_n(t, x_1, \ldots, x_n)$$

Where, $x_i(t)$ are dep. variables, t is a indep. variables and f_i's are known fn.

Lemma 1 (D'Alembert) :- An nth order ODE is equivalent to a system of n first order ODEs.

Pf :- Consider the nth order eqn :

$$x^{(n)} = G(t, x, x', \ldots, x^{(n-1)}) .$$

We introduce the transformation

$$x_1 = x , \; x_2 = x' , \; x_3 = x'' , \; \ldots , \; x_n = x^{(n-1)}(t)$$

Then, $\quad x_1' = x_2 , \; x_2' = x_3 , \; x_3' = x_4 , \ldots , x_n' = G(t, x_1, x_2, \ldots, x_n)$

So, if $\quad X = (x_1, x_2, \ldots, x_n)^T$, then

$$X' = (x_2, x_3, \ldots, G(t, X))^T$$
$$= F(t, X)$$

This is a system of 1st order ODEs.

Existence & Uniqueness :-

Theorem 1: Consider the IVP :

$$x' = f(t,x) \;,\quad x(t_0) = x_0 . \qquad \text{--- (III)}$$

Assume $f : D \to \mathbb{R}$ satisfies the following conditions:

I) $f(t,x)$ is continuous on D, an open connected set in $\mathbb{R}^2$.

II) $\frac{\partial f}{\partial x}(t,x)$ is continuous on D.

Let, $(t_0, x_0) \in D$ and a, b be two positive constants such that the rectangle

$$R = \{ (t,x) : |t - t_0| \le a ,\ |x - x_0| \le b \} \subseteq D .$$

Let, $M = \max_R |f(t,x)|$ and $h = \min \{ a, \frac{b}{M} \}$. Then the IVP (III) has a unique solution $\phi(t)$ in $|t - t_0| \le h$.

Proof :- We will discuss the proof later.

Lemma 2. Let, D is an open connected set in $\mathbb{R}^2$ and $(t_0, x_0) \in D$.

Assume f is continuous on D.

A continuous function $\phi(t)$ is a solution to (III) on an interval I containing t_0 if and only if

$$\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds \;,\qquad t \in I .$$

Pf :-

Suppose $\phi(t)$ is a soln of the IVP (III) on I.

Then, $$\phi'(t) = f(t, \phi(t)), \quad \forall t \in I.$$

Since, ϕ & f are continuous on I and D respectively, $f(t, \phi(t))$ is cont. on I. So,

$$\phi(t) = \phi(t_0) + \int_{t_0}^{t} f(s, \phi(s))\, ds$$

$$= x_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds.$$

Conversely, let, ϕ is a continuous solution of the integral equation.

Since f is continuous, by fundamental thm of integral calculus,

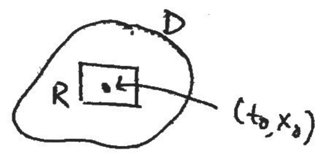
$$\phi'(t) = f(t, \phi(t)), \quad \forall t \in I.$$

Also; $\phi(t_0) = x_0$. So, ϕ is a solution of the IVP. (III).

Uniqueness Theorem :-

Thm 2. Suppose f is cont. on D and $\frac{\partial f}{\partial x}$ is cont on D. Then, the IVP (III) has at most one solution.

Pf :- Since D is an open connected set in $\mathbb{R}^2$, f, $\frac{\partial f}{\partial x}$ can be made continuous in a rectangle $R = \{(t,x) : |t-t_0| \leq a, |x-x_0| \leq b\}$.



Now, since $\frac{\partial f}{\partial x}$ is cont. in R, it is bounded and in turn f is Lipschitz continuous w. r. to x.

Suppose, if possible, $\phi(t)$ and $\psi(t)$ are two solutions of the IVP (III).

Then, by Lemma 1,

$$\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))\,ds$$

& $$\psi(t) = x_0 + \int_{t_0}^{t} f(s, \psi(s))\,ds \qquad \forall t \in I.$$

Now, $$|\phi(t) - \psi(t)| \leq \int_{t_0}^{t} |f(s, \phi(s)) - f(s, \psi(s))|\,ds$$

$$\therefore \quad |\phi(t) - \psi(t)| \leq \alpha \int_{t_0}^{t} |\phi(s) - \psi(s)|\,ds \quad \text{.....(IV)}$$

[Since f is Lipschitz]

Let, $V(t) = \int_{t_0}^{t} |\phi(s) - \psi(s)|\,ds$, so by FTIC,

$$\frac{dV}{dt} = |\phi(t) - \psi(t)| \leq \alpha V(t).$$

$$\Rightarrow \quad \frac{dV}{dt} - \alpha V(t) \leq 0$$

$$\Rightarrow \quad e^{-\alpha(t-t_0)} V(t) \leq V(t_0) = 0. \quad \forall t \geq t_0$$

But, $V(t) \geq 0$ $\forall$ admissible t

Therefore $V(t) = 0$. Hence,

$$0 \leq |\phi(t) - \psi(t)| \leq \alpha V(t) = 0$$

$$\Rightarrow \quad \phi(t) = \psi(t)$$

So, the IVP has a unique solution.

Thm 3 (Gronwall's Inequality):

Suppose p and q are continuous real valued f^n on an interval I, with $q(t) \geq 0$ on I. Let,

$$p(t) \leq c + K \int_{t_0}^{t} q(s)\, p(s)\, ds$$

for all $t \in I$, $t_0 \in I$ and c, K are constants with $K \geq 0$. Then,

$$p(t) \leq c \exp\left(K \int_{t_0}^{t} q(s) ds\right). \quad \forall t \geq t_0.$$

Proof: Let, $F(t) = c + K \int_{t_0}^{t} q(s)\, p(s)\, ds$, then,

$$p(t) \leq F(t) \qquad \forall t \in I. \quad \cdots (I)$$

Now, by fundamental theorem of calculus,

$$F'(t) = K q(t)\, p(t) \leq K q(t) F(t), \quad \text{by (I)}.$$

$$\Rightarrow \quad F'(t) - K q(t) F(t) \leq 0.$$

Now,

$$\frac{d}{dt}\left(\exp\left(-\int_{t_0}^{t} K q(s)\, ds\right) F(t)\right)$$

$$= \underbrace{\exp\left(-K \int_{t_0}^{t} q(s) ds\right)}_{> 0} \left[F'(t) - K q(t) F(t)\right] \leq 0$$

Therefore, $\exp\left(-K \int_{t_0}^{t} q(s) ds\right) F(t)$ is a decreasing function,

so that,

$$\exp\left(-K \int_{t_0}^{t} q(s)\, ds\right) F(t) \leq F(t_0) = c \quad \text{for } t \geq t_0.$$

i.e. $$F(t) \leq c \exp\left(K \int_{t_0}^{t} q(s)\, ds\right). \quad \cdots (II)$$

(I) & (II) $\Rightarrow$ $$p(t) \leq c \exp\left(K \int_{t_0}^{t} q(s)\, ds\right) \quad \text{for } t \geq t_0.$$

Cor: If $p(t) \geq 0$ $\forall\, t \in I$ and $c = 0$ in Thm 3, then,

$p(t) \equiv 0$ $\forall\, t \geq t_0$.

NOTE:- In proof of Thm 2, $|\phi(t) - \psi(t)| \equiv 0$ by the Cor.

Problems

1. **General Form:-**

Suppose p, q and f are continuous on I with $q(t) \geq 0$ on I. Let,

$$p(t) \leq f(t) + \int_{t_0}^{t} q(s)\, p(s)\, ds \quad \text{for all } t \in I \text{ and}$$

$t_0 \in I$. Then,

$$p(t) \leq f(t) + \int_{t_0}^{t} f(s)\, q(s)\, e^{\int_s^t q(\tau)\, d\tau}\, ds, \quad t \geq t_0.$$

2. The conditions of the existence - uniqueness theorem are sufficient but not neccessary.

Exm: $y' = \sqrt{y} + 1$, $y(0) = 0$, $x \in [0,1]$.

In this case, $f(x,y) = \sqrt{y} + 1$ is not Lipschitz near origin.

But it has a unique solution, i.e. $(\sqrt{y}+1)e^{-\sqrt{y}} = e^{-x/2}$

$\left(z(x) = (\sqrt{y_1} - \sqrt{y_2})^2 \right)$

3. $y' = \frac{1}{y^2}$, $y(x_0) = 0$. $f(x,y) = \frac{1}{y^2}$ is discontinuous at $y = 0$

However $y(x) = [3(x - x_0)]^{1/3}$ is the unique soln. Does it contradict the Existence - Uniqueness thm?

4. If $g : [0,T] \to \mathbb{R}$ is cont. and if there are non-negative constants c, B, K such that

$$g(t) \leq c + Bt + K\int_0^t g(s)\, ds, \quad 0 \leq t \leq T.$$

then,

$$g(t) \leq c e^{Kt} + B\, \frac{e^{Kt} - 1}{K}, \quad 0 \leq K \leq T.$$

5. Show that if $g : [0,T] \to \mathbb{R}$ is continuous and if there are non-neg. constants c, M, K with $M < K$ such that

$$g(t) \le c(e^{Mt} - 1) + K \int_0^t g(s)\,ds \quad , \quad 0 \le t \le T$$

then,
$$g(t) \le \frac{c}{K/M - 1} \left(e^{Kt} - e^{Mt} \right) \quad , \quad 0 \le t \le T.$$

6. Show that $\frac{dx}{dt} = -\sqrt{x}$, $x(0) = b$, $b > 0$, $0 \le t < \infty$

has unique solution. [Hint: $z(t) = (x(t) - y(t))^2$.]

7. Let, $u(x)$ be a non-negative continuous function in the interval $|x - x_0| \le a$ and $c \ge 0$ be a constant, and

$$u(x) \le \left| \int_{x_0}^{x} c\, u^{\alpha}(t)\,dt \right| \quad , \quad 0 < \alpha < 1.$$

Show that for all x in $|x - x_0| \le a$,

$$u(x) \le \left\{ c(1-\alpha)\,|x - x_0| \right\}^{1/1-\alpha}.$$

Picard - Lindelöf Theorem :-

Thm 4. Let, $f : D \to \mathbb{R}$ satisfies the following conditions:

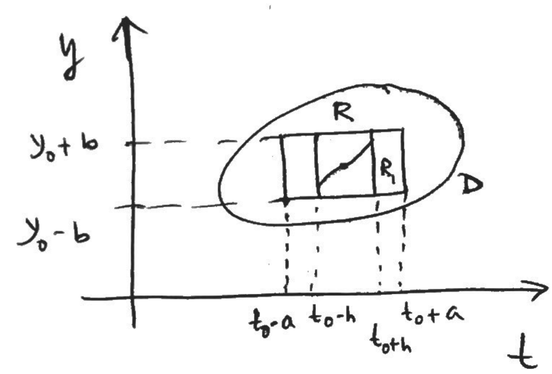
i) $f(t, y)$ is continuous on D

ii) f is Lipschitz continuous w.r.to y on D with Lip. Constant $\alpha > 0$.

Let, $(t_0, y_0) \in D$ and $a \,\&\, b \in \mathbb{R}$ such that

$$R = \{(t, y) : |t - t_0| \le a, \ |y - y_0| \le b\}$$ is a subset of D. Let, $M = \max_R |f(t,y)|$ and $h = \min\{a, \frac{b}{M}\}$.

Then $y' = f(t, y), \ y(t_0) = y_0$ has a unique solution in $|t - t_0| \le h$.



Proof :- We shall prove the theorem by the method of successive approximations.

Let, $R_1 = \{(t, y) : |t - t_0| \le h, \ |y - y_0| \le b\}$. Then,

$R_1 \subseteq R$.

For $|t - t_0| \le h$, define $\phi_0(t) := y_0$ and

$$\phi_m(t) := y_0 + \int_{t_0}^{t} f(t, \phi_{m-1}(t))\, dt, \quad m = 1, 2, \ldots$$

We do the following steps:

S1. $\{\phi_n\}_n$ are well-defined on $|t - t_0| \le h$ and are continuously differentiable.

S2. $\{\phi_n\}_n$ converges uniformly to a continuous function ϕ on $|t - t_0| \le h$.

S3. ϕ is a solution the ~~the~~ ODE in $|t - t_0| \le h$ with $\phi(t_0) = y_0$.

S1. $\phi_0(t) = y_0$. Now, f and ϕ_0 being continuous in R_1,

$$\phi_1(t) = y_0 + \int_{t_0}^{t} f(t, \phi_0(t))\, dt$$ is continuous by F.T.C.,

and $$|\phi_1(t) - y_0| \le \int_{t_0}^{t} |f(t, \phi_0(t))|\, dt$$
$$\le M|t - t_0| \le Mh \le b$$

So, $(t, \phi_1(t)) \in R_1$ for $|t - t_0| \le h$.

Suppose the result is true for $m \in \mathbb{N}$. ie. $\phi_m(t)$ is well-defined, continuous and $(t, \phi_m(t)) \in R_1$ for $|t-t_0| \le h$.

Now, $$\phi_{m+1}(t) = y_0 + \int_{t_0} f(s, \phi_m(s))\, ds, \quad |t - t_0| \le h.$$

Since ϕ_m and f are continuous, $\phi_{m+1}(t)$ is cont.

Now, $$|\phi_{m+1}(t) - y_0| \le \int_{t_0}^{t} |f(s, \phi_m(s))|\, ds \le M|t - t_0|$$
$$\le Mh \le b,$$

So, $(t, \phi_{m+1}(t)) \in R_1$ for $|t - t_0| \le h$.

Hence, by Mathematical Induction, the result holds for all $n \in \mathbb{N}$.

S2. To show $\phi_n(t) \rightrightarrows \phi(t)$ in $|t - t_0| \le h$

Note that
$$\phi_n(t) = y_0 + \sum_{i=1}^{n} [\phi_i(t) - \phi_{i-1}(t)]$$

Therefore, $\phi_n(t)$ is the nth partial sum of the series

$$y_0 + \sum_{i=1}^{\infty} [\phi_i(t) - \phi_{i-1}(t)] \quad \cdots (\star)$$

So, proving the uniform convergence of $\{\phi_n(t)\}_n$ is equivalent to proving the uniform convergence of the series $(\star)$.

First,
$$|\phi_1(t)-\phi_0(t)| = \left|\int_{t_0}^{t} f(s,\phi_0(s))\,ds\right|$$
$$\leq M|t-t_0|$$

We shall prove by Mathematical Induction, that,
$$|\phi_m(t)-\phi_{m-1}(t)| \leq M\frac{\alpha^{m-1}|t-t_0|^m}{m!}, \quad m\in\mathbb{N}. \quad \ldots (**)$$

The result is clearly true for $m=1$.

Assume it is true for $m=K-1$, i.e.
$$|\phi_{K-1}(t)-\phi_{K-2}(t)| \leq M\frac{\alpha^{K-2}|t-t_0|^{K-1}}{(K-1)!}.$$

Now,
$$|\phi_K(t)-\phi_{K-1}(t)| \leq \int_{t_0}^{t} |f(s,\phi_{K-1}(s))-f(s,\phi_{K-2}(s))|\,ds$$
$$\leq \alpha\int_{t_0}^{t} |\phi_{K-1}(s)-\phi_{K-2}(s)|\,ds$$
$$\leq \frac{M\alpha^{K-1}}{(K-1)!}\int_{t_0}^{t} |s-t_0|^{K-1}\,ds$$
[Prove seperately for (t_0,t_0+h) and (t_0-h,t_0)]
$$= \frac{M\alpha^{K-1}}{K!}|t-t_0|^K.$$

Hence $(**)$ holds for all $m\in\mathbb{N}$.

Now,
$$\sum_{m=1}^{\infty} M\frac{\alpha^{m-1}|t-t_0|^m}{m!} \leq \frac{M}{\alpha}\sum_{m=1}^{\infty}\frac{(\alpha h)^m}{m!} = \frac{M}{\alpha}e^{\alpha h} < \infty.$$

Therefore, by Weierstrass M-test, the Series (A) Converges absolutely and uniformly in $|t-t_0|\leq h$. So, the partial sum sequence $\{\phi_n(t)\}_n$ converges absolutely & uniformly in $|t-t_0|\leq h$. Suppose
$$\phi_n(t) \rightrightarrows \phi(t) \text{ in } |t-t_0|\leq h.$$

Since, each $\Phi_n(t)$ is continuous, Φ is continuous.

S3. Also, $\forall n \in \mathbb{N}$

$$|\Phi_n(t) - y_0| \le b \quad \text{for } |t-t_0| \le h, \text{ so}$$

$$|\Phi(t) - y_0| \le b. \quad \text{Hence, } (t, \Phi(t)) \in R,$$

Now,

$$\Phi(t) = \lim_{n\to\infty} \Phi_n(t) = \lim_{n\to\infty}\left[y_0 + \int_{t_0}^{t} f(s, \Phi_{n-1}(s))\, ds\right]$$

$$= y_0 + \int_{t_0}^{t} \lim_{n\to\infty} f(s, \Phi_{n-1}(s))\, ds.$$

[$\Phi_n \rightrightarrows \Phi$ and f Lipschitz $\Rightarrow$ $f(s,\Phi_n(s))$ are uniformly convergent] (Check)!

$$\therefore \quad \Phi(t) = y_0 + \int_{t_0}^{t} f(s, \Phi(s))\, ds, \quad |t-t_0| \le h$$

By Lemma 2, Φ satisfies the IVP:

$$y' = f(t,y), \quad y(t_0) = y_0.$$

This proves the existence of a solution in $|t-t_0| \le h$.

The uniqueness is guranteed by Thm 2.

Note:

$$\lim_{n\to\infty} \int_{t_0}^{t} f(s, \Phi_n(s))\, ds = \int_{t_0}^{t} f(s, \Phi(s))\, ds$$

because,

$$\left|\int_{t_0}^{t} f(s,\Phi_n(s))\, ds - \int_{t_0}^{t} f(s,\Phi(s))\, ds\right|$$

$$\le \int_{t_0}^{t} \alpha\, |\Phi_n(s) - \Phi_n(s)|\, ds$$

$$\le \alpha \sup_{[t_0,t]} |\Phi_n - \Phi_n|\, |t-t_0|$$

$$\le \alpha h \sup_{|t-t_0|\le h} |\Phi_n - \Phi| < \frac{\epsilon}{\alpha h}\cdot \alpha h = \epsilon \quad \forall n \ge N_0$$

where, $|\Phi_n(t) - \Phi(t)| < \dfrac{\epsilon}{\alpha h}$ $\quad \forall t \in [t_0-h, t_0+h)$, $\forall n \ge N_0$