Defn. (Uniformly Bounded) Let, T be a collection of functions on $D \subseteq \mathbb{R}$. T is said to be uniformly bounded on D if $\exists\, B > 0$ such that

$$|f(x)| \leq B \quad \forall x \in D \text{ and } \forall f \in T.$$

Defn. (Equicontinuous) The collection T is said to be equicontinuous on D if for given $\epsilon > 0$, $\exists\, \delta > 0$ s.t.

$$|f(x) - f(y)| < \epsilon \quad \text{when } |x - y| < \delta,$$
for $x, y \in D$ and for $f \in T$.

Arzela - Ascoli Theorem :-

On a compact set D, let $T = \{f\}$ be an infinite, uniformly bounded, equicontinuous set of functions. Then, T contains a sequence $\{f_n\}_n$ of functions, converges uniformly to a continuous function f_0.

Theorem 5 (Cauchy - Peano Theorem)

Let, $R = \{(t,x) : |t - t_0| \leq a,\ |x - x_0| \leq b\}$ and $f(t,x)$ is continuous on R. Then, the IVP

$$x' = f(t,x), \quad x(t_0) = x_0 \quad \cdots (*)$$

has at least one solution for $|t - t_0| \leq h$, where $h = \min\left\{a, \frac{b}{M}\right\}$ with $M = \max\limits_{R} |f(t,x)|$.

Proof :- We show the existence of a solution on $[t_0, t_0 + h]$. A similar argument works for $[t_0 - h, t_0]$.

Since $h > 0$, $\exists\, N \in \mathbb{N}$ such that

$$\frac{1}{N} < h.$$

Therefore $\left\{\frac{1}{n}\right\}_n \subseteq [0, h]$, $\forall n \geq N$.

Now, define

$$\phi_{\frac{1}{n}}(t) = \begin{cases} x_0, & t_0 - \frac{1}{n} < t \le t_0 \\ x_0 + \int_{t_0}^{t} f(s, \phi_{1/n}(s - \frac{1}{n}))\, ds, & t_0 \le t \le t_0 + h. \end{cases}$$

S1. $\phi_{1/n}(t)$ is well-defined in $[t_0, t_0+h]$

S2. $T = \{ \phi_{1/n}(t) : t \in [t_0, t_0+h],\ n \ge N \}$ is equicontinuous and uniformly bounded in $[t_0, t_0+h]$.

S3. $\exists$ a sequence $\{\phi_{1/n_k}\}_k$ converging uniformly to $\phi(t)$, on $[t_0, t_0+h]$. ϕ is a solution to $(*)$.

S1. For $t \in [t_0, t_0+h]$,

$$|\phi_{1/n}(t) - x_0| = \left| \int_{t_0}^{t} f(s, \phi_{1/n}(s - \tfrac{1}{n}))\, ds \right|$$

$$\le M|t - t_0| \le Mh \le b.$$

So, each $\phi_{1/n}(t)$ is well-defined, $n \ge N$.

S2. Consider the collection of functions

$$T = \{ \phi_{1/n}(t) : t \in [t_0, t_0+h],\ n \ge N \}.$$

We'll show

$$|\phi_{1/n}(t) - x_0| \le b \quad \text{for } t \in [t_0, t_0+h] \quad \forall\, n \ge N. \quad \cdots \text{(A)}$$

For $t \in [t_0, t_0 + \frac{1}{n}]$,

$$|\phi_{1/n}(t) - x_0| = \left| \int_{t_0}^{t} f(s, \phi_{1/n}(s - \tfrac{1}{n}))\, ds \right|$$

$$\le M|t - t_0| \le Mh \le b$$

Assume (A) is true for $t \in [t_0 + (k-1)\frac{1}{n}, t_0 + \frac{k}{n}] \cap [t_0, t_0+h]$

Then, $(t, \phi_{1/n}(t)) \in R_1$

Let, $t \in \left[t_0 + \frac{k}{n}, t_0 + \frac{(k+1)}{n}\right] \cap [t_0, t_0 + h]$.

Then,

$$|\phi_{1/n}(t) - x_0| = \left| \int_{t_0}^{t} f(s, \phi_{1/n}(s - \tfrac{1}{n}))\, ds \right|$$

$$\leq M|t - t_0| \leq Mh \leq b.$$

So, by Mathematical Induction, (A) is true for all $t \in [t_0, t_0 + h]$, $n \geq N$.

We now will show, T is equicontinuous.

Let, $\epsilon > 0$, $t_1, t_2 \in [t_0, t_0 + h]$. such that $|t_1 - t_2| < \frac{\epsilon}{M} (= \delta)$.

Then,

$$|\phi_{1/n}(t_1) - \phi_{1/n}(t_2)| = \left| \int_{t_0}^{t_1} f(s, \phi_{1/n}(s - \tfrac{1}{n}))\, ds - \int_{t_0}^{t_2} f(s, \phi_{1/n}(s - \tfrac{1}{n}))\, ds \right|$$

$$= \left| \int_{t_1}^{t_2} f(s, \phi_{1/n}(s - \tfrac{1}{n}))\, ds \right|$$

$$\leq M|t_2 - t_1| < \epsilon. \qquad \forall\, n \geq N.$$

This means, T is equicontinuous on $[t_0, t_0 + h]$.

S3. So by Arzela-Ascoli theorem, $\exists$ a sequence $\{\phi_{1/n_k}\}_k$ of T, uniformly converges to, say, $\phi(t)$ in $[t_0, t_0 + h]$. i.e.

$$\lim_{k \to \infty} \phi_{1/n_k}(t) = \phi(t).$$

Since each ϕ_{1/n_k} is continuous on $[t_0, t_0 + h]$, so is ϕ.

Now,

$$\phi_{1/n_k}(t) = x_0 + \int_{t_0}^{t} f(s, \phi_{1/n_k}(s - 1/n_k))\, ds$$

Take limit on both sides as $k \to \infty$.

then for $t \in [t_0, t_0+h]$,

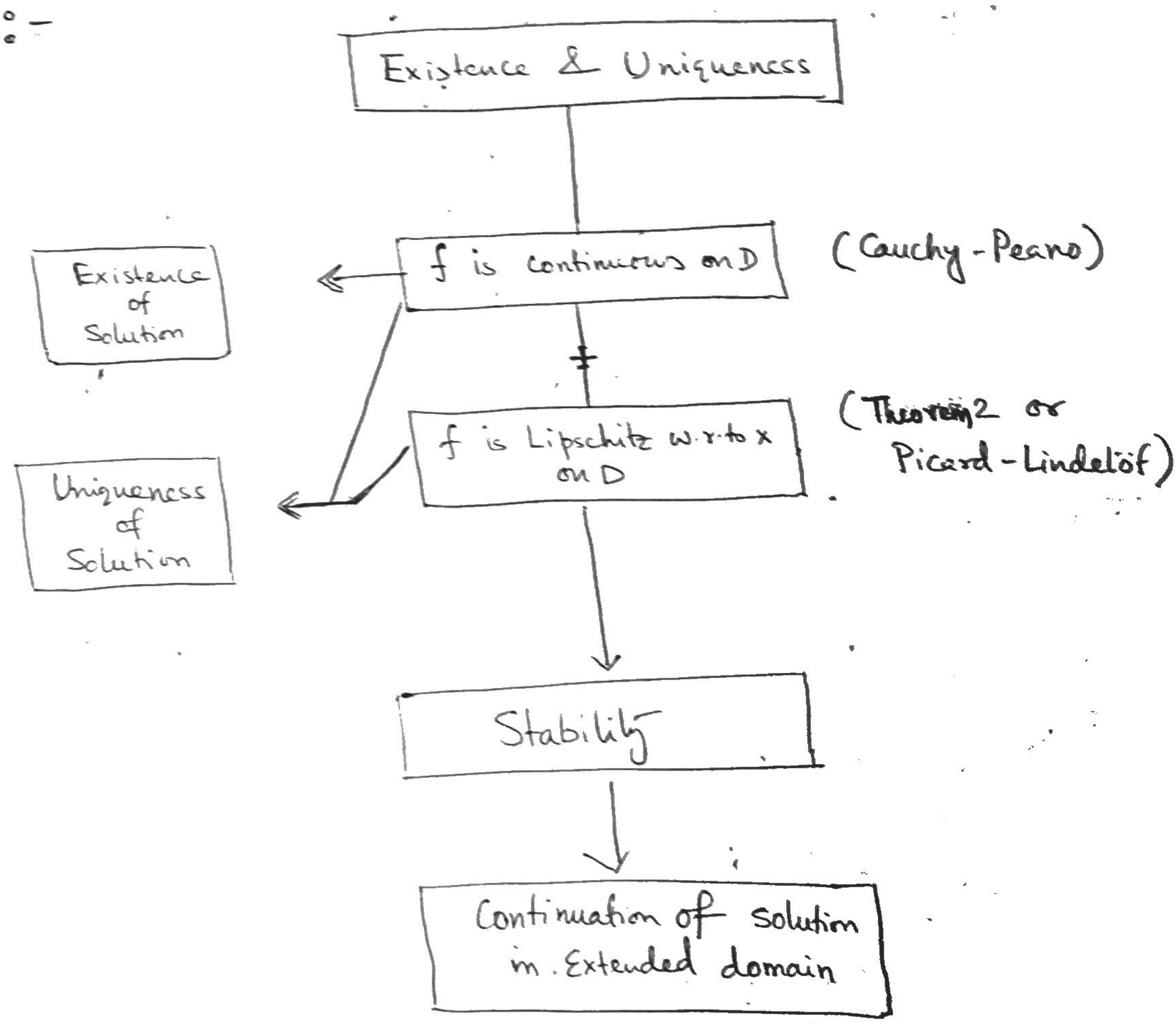
$$\phi(t) = x_0 + \lim_{k\to\infty} \int_{t_0}^{t} f\left(s, \phi_{1/n_k}(s - 1/n_k)\right) ds$$

$$= x_0 + \int_{t_0}^{t} f\left(s, \lim_{k\to\infty} \phi_{1/n_k}\left(s-\frac{1}{n_k}\right)\right) ds \qquad \left[\begin{array}{l} \because f \text{ is uniformly} \\ \text{cont on} \\ \text{compact} \\ [t_0-h, t_0+h] \text{ \& } \\ \phi_{1/n_k} \rightrightarrows \phi \end{array}\right]$$

$$\therefore \phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds$$

So, by Lemma 2, $\phi(t)$ is a solution of the IVP (*). $\left[\because \phi_{1/n_k} \text{ is cont.}\right]$

NOTE :-



Exm: $y' = y, \quad y(0) = 1.$

Picard iteration.

$$\phi_0(x) = y_0 = 1.$$

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$= 1 + \int_0^x 1\, dx = 1 + x$$

$$\phi_2(x) = y_0 + \int_0^x f(x, \phi_1(x))\, dx$$

$$= 1 + \int_0^x (1+x)\, dx = 1 + x + \frac{x^2}{2}$$

$$\phi_3(x) = 1 + \int_0^x \left(1 + x + \frac{x^2}{2}\right) dx = 1 + x + \frac{x^2}{2!} + \frac{x^2}{3!}$$

$\vdots$

$$\phi_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} \rightarrow e^x \text{ as } n \to \infty.$$

The given IVP has unique solution : $\phi(x) = e^x$.

Theorem 6 (Continuous dependence on Initial Data)

Let, f be a continuous function on the rectangle

$R = \{(t,x) : |t - t_0| \le a, \ |x - x_0| \le b\}$ and satisfies Lipschitz condition. If in the ~~init~~ IVP

$$x' = f(t,x), \quad x(t_0) = x_0 \quad \dots (*)$$

the initial data changes by a small amount, say $|x_0 - x_0^*| < \delta$.

then, the solution changes accordingly, i.e.

$$|\phi(t) - \psi(t)| < \epsilon.$$

Proof :- Since $\phi(t)$ and $\psi(t)$ are solutions to $(*)$, we have by Lemma 2,

$$\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds, \quad |t - t_0| \le a$$

and $$\psi(t) = x_0^* + \int_{t_0}^{t} f(s, \psi(s))\, ds, \quad |t - t_0| \le a.$$

Let, $\epsilon > 0$ and $\delta = \epsilon e^{-a\alpha}$. Then, for $|t-t_0| \le a$

$$|\phi(t) - \psi(t)| \le |x_0 - x_0^*| + \int_{t_0}^{t} |f(s,\phi(s)) - f(s,\psi(s))|\, ds$$

$$\therefore\ |\phi(t) - \psi(t)| \le |x_0 - x_0^*| + \alpha \int_{t_0}^{t} |\phi(s) - \psi(s)|\, ds$$

Now by Gronwall's inequality, we have

$$|\phi(t) - \psi(t)| \le |x_0 - x_0^*| \exp\left(\alpha|t-t_0|\right)$$
$$< \delta e^{\alpha a} = \epsilon.$$

when $|x_0 - x_0^*| < \delta$.

This completes the result.

Continuation of Solution :-

EXM: $x' = x^2$, $x(0) = 1$ .. (*)

Consider $R = \{ (t,x) : |t-0| \le a$ and $|x-1| \le b \}$

Now, $f(t,x) = x^2$ is continuous & Lipschitz w.r. to x in R.

So, the IVP (*) has a unique soln

$$x(t) = \frac{1}{1-t}, \quad \text{for } |t-0| \le h = \min\left\{ a, \frac{b}{M} \right\}$$

with $M = \max_R |x^2| = (1+b)^2$

But, $(1-b)^2 \ge 0 \implies 1 + 2b + b^2 \ge 4b$

$$\implies \frac{b}{(1+b)^2} \le \frac{1}{4}.$$

So, irrespective of a, we can take $h = \frac{1}{4}$.

Hence, the solution is: $x(t) = \frac{1}{1-t}$, $t \in [-\frac{1}{4}, \frac{1}{4}]$.

But, we have seen that, $x(t) = \frac{1}{1-t}$ is a solution in $(-\infty, 1)$.

We need to maximize the interval of existence of solution.

Lemma 3. Let, $f(t,x)$ be continuous on D & f is bounded on D. Let, the IVP

$$x' = f(t,x), \quad x(t_0) = x_0$$

has a solution in (α, β). Then, $\phi(\alpha+)$ and $\phi(\beta-)$ exist.

Proof :-

For $\alpha < t_1 < t_2 < \beta$, wehave,

$$|\phi(t_2) - \phi(t_1)| \leq \int_{t_1}^{t_2} |f(s, \phi(s))\, ds$$

$$\leq M\, |t_2 - t_1|$$

As $t_1, t_2 \to \alpha+$, $\phi(t_2) - \phi(t_1) \to 0$. So by Cauchy criterion,

$$\lim_{t \to \alpha+} \phi(t) = \phi(\alpha+) \text{ exists.}$$

ll.ly, $\phi(\beta-)$ exists.

Theorem 7. Let, $f(t,x)$ be continuous and bounded on D and ϕ is a soln in (α,β).
Let, $(\beta, \phi(\beta-)) \in D$ $[$or $(\alpha, y(\alpha+)) \in D]$. Then the solution ϕ of the IVP

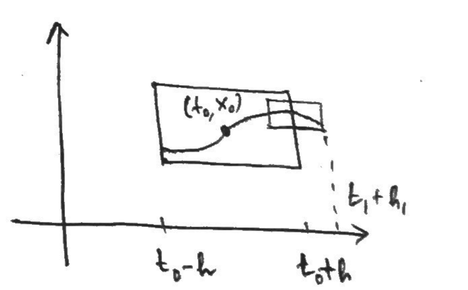
$$x' = f(t,x), \quad x(t_0) = x_0$$

can be extended over $(\alpha, \beta + h_1]$ (or $[\alpha - h_1, \beta)$) for some $h_1 > 0$.

Hint : By Lemma 3, $\phi(\alpha+)$ and $\phi(\beta-)$ exist.

Define $\phi_1(t)$ in $(\alpha, \beta]$ as:

$$\phi_1(t) = \begin{cases} \phi(t), & t \in (\alpha,\beta) \\ \phi(\beta-), & t = \beta. \end{cases}$$



We have $\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds$, $t \in (\alpha, \beta)$.

$$\phi_1(\beta) = \phi(\beta-)$$
$$= x_0 + \int_{t_0}^{\beta} f(s, \phi(s))\,ds$$
$$= x_0 + \int_{t_0}^{\beta} f(s, \phi_1(s))\,ds$$

ϕ_1 is a continuation of ϕ in $(\alpha, \beta]$. Let, $\phi_2(t)$ be a solution of

$$x' = f(t,x), \quad x(\beta) = \phi(\beta-).$$

in some $[\beta, \beta+h_1]$. Then we can define: ϕ_3 in $(\alpha, \beta+h_1]$ as follows:

$$\phi_3(t) = \begin{cases} \phi_1(t), & t \in (\alpha, \beta] \\ \phi_2(t), & t \in [\beta, \beta+h_1]. \end{cases}$$

Note, $\phi_3(t_0) = \phi_1(t_0) = \phi(t_0) = x_0$. and for $t \in [\beta, \beta+h_1]$

$$\phi_2(t) = \phi(\beta-) + \int_{\beta}^{t} f(s, \phi_2(s))\,ds$$
$$= x_0 + \int_{t_0}^{\beta} f(s, \phi_1(s))\,ds + \int_{\beta}^{t} f(s, \phi_2(s))\,ds$$
$$\therefore \phi_3(t) = x_0 + \int_{t_0}^{t} f(s, \phi_3(s))\,ds$$

Hence, by Lemma 2, ϕ_3 is a solution of

$$x' = f(t,x), \quad x(t_0) = x_0 \quad \text{in } (\alpha, \beta+h_1].$$

This completes the result.

Exm :- $\quad x' = \frac{\pi}{2}(1+x^2), \quad x(0) = 0.$

$$\phi(t) = \tan\left(\frac{\pi}{2}t\right) \quad t \in [-h, h] \text{ where}$$
$$h = \min\left\{a, \frac{2b}{\pi(1+b^2)}\right\} \leq \frac{1}{\pi}.$$

But the maximum interval would be $(-1, 1)$.

H.W. 1. Show that the solution of the problem $y' = y^2$, $y(0) = 2$ is extendable to the maximum interval $(-\infty, \frac{1}{2})$.

2. Show that the solution of $y' = 2xy^2$, $y(0) = 1$ exists only in the interval $|x| < 1$.