Second Order Equations

BCM ODEs

Defn:- A 2nd order linear differential equation is of the form:

$$\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} + q(t)\,x = f(t). \quad \cdots (I)$$

where $p(t)$, $q(t)$ and $f(t)$ are given functions on $I \subseteq \mathbb{R}$.

We denote $\mathcal{L}(x) \equiv \frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} + q(t)\,x$, then the ODE becomes:

$$\mathcal{L}(x) = f(t). \quad \cdots\cdots (II).$$

$\Rightarrow$ The eqn (II) is called homogeneous if $f(t) = 0 \ \forall t \in I$, else, it is called non-homogeneous.

Existence & Uniqueness Theorem:- Let, p, q, f be continuous in a compact interval $I \subseteq \mathbb{R}$. For any $t_0 \in I$ and constants α_1, α_2, $\exists$ a unique solution for the IVP:

$$\begin{cases} \mathcal{L}(x) = f(t) \\ x(t_0) = \alpha_1, \ x'(t_0) = \alpha_2 \end{cases}$$

Defn:- An operator $\mathcal{L} \equiv D^n + a_{n-1}(t)D^{n-1} + \cdots + a_1 D + a_0$ is said to be a linear operator if for any n times differentiable functions y_1 & y_2 and constants c_1, c_2, we have

$$\mathcal{L}\big(c_1 y_1(t) + c_2 y_2(t)\big) = c_1\mathcal{L}(y_1) + c_2\mathcal{L}(y_2). \quad \text{where,}$$

$$D \equiv \frac{d}{dt}.$$

(Superposition Principle) Theorem 8. Let, $\phi_1(t)$ and $\phi_2(t)$ are solutions for $\mathcal{L}(x) = f_1$ and $\mathcal{L}(x) = f_2$ respectively. Then, $\alpha_1\phi_1(t) + \alpha_2\phi_2(t)$ is a solution of

$$\mathcal{L}(x) = \alpha_1 f_1 + \alpha_2 f_2, \quad \alpha_1, \alpha_2 \in \mathbb{R}.$$

The Superposition principle is NOT true for non-linear equations:

EXM:

$$x^3 y'' - y y' = 0.$$

$y_1 = 1$ and $y_2 = x^2$ are two solutions of the ODE.

But, their linear combination $4 + 3x^2$ is not a soln.

NOTE:- If $f_1 = f_2 = 0$ in Thm 8., then $c_1\phi_1 + c_2\phi_2$ is also a soln of $\mathcal{L}(x) = 0$.

This is not true for non-homogeneous eqns:

EXM:-

$$y'' - 9y = 18.$$

$y_1 = 4e^{3x} - 2$ and $y_2 = e^{3x} - 2$ are two solns.

But, their sum $5e^{3x} - 4$ is not a soln of $y'' - 9y = 18$.

Defn:- The functions $f_1, f_2, \ldots, f_n$ are said to be linearly dependent on I, if $\exists$ constants $c_1, c_2, \ldots, c_n$, not all zeros, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) \equiv 0 \quad \forall x \in I.$$

The set of functions is said to be linearly independent if it is not linear dependent.

EXM:- $f_1 = 1$, $f_2 = x$, $f_3 = x^2$ are L.I.

EXM:- $f_1(x) = \begin{cases} 0, & x \in [0, \frac{1}{2}) \\ x, & x \in [\frac{1}{2}, 1] \end{cases}$, $f_2(x) = \begin{cases} 0 & x = 0 \\ x^2, & \text{else} \end{cases}$, $f_3(x) = (1 + x)$

are L.I.

Exm. $f_1 = x$, $f_2 = 2x$, $f_3(x) = x^2$ are L.D.

Exm: $f_1 = x$, $f_2 = |x|$ are LI.

Definition :- The ... wronskian of two differentiable functions $\phi_1(x)$ & $\phi_2(x)$ on an interval I is defined by

$$W(\phi_1, \phi_2; x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{vmatrix} = \phi_1 \phi_2' - \phi_2 \phi_1'.$$

Theorem 9. Let, ϕ_1 & ϕ_2 be two solutions of $\mathcal{L}(y) = 0$ on I. Then, ϕ_1 & ϕ_2 are L.I iff $W(\phi_1, \phi_2; x) \neq 0 \ \forall x \in I$.

Proof :- Let, $W(\phi_1, \phi_2; x) \neq 0 \quad \forall x \in I$.

Consider $c_1 \phi_1(x) + c_2 \phi_2(x) = 0. \quad \forall x \in I$.

So, we have,

$$c_1 \phi_1'(x) + c_2 \phi_2'(x) = 0. \quad \forall x \in I.$$

Let, $x_0 \in I$, then

$$\phi_1(x_0) c_1 + \phi_2(x_0) c_2 = 0$$
$$\phi_1'(x_0) c_1 + \phi_2'(x_0) c_2 = 0.$$

Now, since $W(\phi_1, \phi_2; x_0) \neq 0$, so by Cramer's Rule, the homogeneous system has unique solⁿ, i.e.

$$c_1 = 0 = c_2.$$

Hence, ϕ_1 & ϕ_2 are L.I.

Conversely, let, ϕ_1, ϕ_2 are L.I. in I.

If possible, let, $W(\phi_1, \phi_2; x_0) = 0$ for $x_0 \in I$.

Then, the system

$$\phi_1(x_0) c_1 + \phi_2(x_0) c_2 = 0$$
$$\phi_1'(x_0) c_1 + \phi_2'(x_0) c_2 = 0$$

has a non-trivial solution (c_1^*, c_2^*). i.e.

$$|c_1^*| + |c_2^*| \neq 0.$$

Consider, $X(t) = c_1^* \phi_1(x) + c_2^* \phi_2(x)$. and the

IVP:

$$\mathcal{L}(y)=0, \quad y(x_0)=0, \quad y'(x_0)=0.$$

Then, clearly $X(x)$ is a solution of the IVP.

Also, $y(x) \equiv 0$ is another solution. But, by uniqueness,

$$\therefore \quad X(x) \equiv 0 \quad \forall x \in I.$$

$$\Rightarrow \quad c_1^* \phi_1(x) + c_2^* \phi_2(x) = 0 \quad \forall x \in I.$$

$$\Rightarrow \quad \phi_1, \phi_2 \text{ are L.D., a contradiction.}$$

Therefore, $W(\phi_1, \phi_2; x) \neq 0 \quad \forall x \in I$.

<u>NOTE</u> :- $W(\phi_1, \phi_2; x) \neq 0 \quad \forall x \in I \Rightarrow \phi_1, \phi_2$ are L.I.

but, ϕ_1, ϕ_2 are LI $\not\Rightarrow W(\phi_1, \phi_2; x) \neq 0$. To satisfy Theorem 9, ϕ_1, ϕ_2 have to solve $\mathcal{L}(y)=0$.

<u>EXM</u>: $\phi_1(x) = x, \quad \phi_2(x) = x^2, \quad x \in [-1, 1]$.

We know ϕ_1, ϕ_2 are L.I. But,

$$W(\phi_1, \phi_2; 0) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2\Big|_{x=0} = 0.$$

<u>Theorem 10.</u> (Abel Theorem) Let, ϕ_1 & ϕ_2 be two solutions of the diff. eqn $\mathcal{L}(y)=0$. Then the wronskian satisfies a 1st order differential equation.

<u>Proof</u>:- Let, ϕ_1 & ϕ_2 be two solutions of

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \quad \cdots (1)$$

We have

$$W(\phi_1, \phi_2; x) = \phi_1 \phi_2' - \phi_2 \phi_1'$$

So,

$$\frac{dW}{dx} = \phi_1 \phi_2'' - \phi_2 \phi_1''$$
$$= \phi_1(x)\left[-p(x)\phi_2'(x) - q(x)\phi_2(x)\right] - \phi_2(x)\left[-p\phi_1' - q\phi_1\right]$$

$$= p(x)\left[\phi_2\phi_1' - \phi_1\phi_2'\right]$$

$$= -p(x)\, W(\phi_1, \phi_2; x)$$

$$\therefore \quad \frac{dW}{dx} + p(x)\, W = 0 \qquad \cdots (11)$$

So, $W(x)$ satisfies the ODE (11).

Theorem 11. Let, $\phi_1(x)$ & $\phi_2(x)$ be two solutions of $\mathcal{L}(y) = 0$ on I. Then, $W(\phi_1, \phi_2; x)$ is either identically zero or is never zero on I.

Proof :- We have from Theorem 10,

$$W' + p(x) W = 0$$

$$\Rightarrow \quad \frac{d}{dx}\left[W(x) \exp\left(\int_{x_0}^{x} p(s)\, ds\right)\right] = 0$$

$$\Rightarrow \quad W(x) \exp\left(\int_{x_0}^{x} p(s)\, ds\right) - W(x_0) = 0,$$

after integrating over x_0 to x.,

$$\Rightarrow \quad W(\phi_1, \phi_2; x) = W(x_0) \exp\left(-\int_{x_0}^{x} p(s)\, ds\right)$$

We now have two possibilities:

$W(x_0) = 0$ and $W(x_0) \neq 0$.

Accordingly, $W(\phi_1, \phi_2; x) = 0$ or $W(\phi_1, \phi_2; x) \neq 0$. on I.

NOTE :- Thm 11 is not true if ϕ_1 & ϕ_2 are not solutions to some $\mathcal{L}(y) = 0$.

Exm: $\phi_1(x) = \sin x$, $\phi_2(x) = x \sin x$, $W(\phi_1, \phi_2; x) = \sin^2 x$

Theorem 12. Let, ϕ_1 & ϕ_2 be two L.I solutions of $\mathcal{L}(y)=0$ on I. Then every solution of $\mathcal{L}(y)=0$ can be expressed uniquely as

$$\phi(x) = c_1\phi_1(x) + c_2\phi_2(x) \quad \forall x \in I.$$

Proof:- Let, $\phi(x)$ be any solution of $\mathcal{L}(y)=0$ on I.

Let, $x_0 \in I$ and $\phi(x_0)=\alpha$, $\phi'(x_0)=\beta$.

Since ϕ_1 and ϕ_2 are L.I., $W(\phi_1,\phi_2;x_0) \neq 0$.

So, the system of equations:

$$\phi_1(x_0)c_1 + \phi_2(x_0)c_2 = \alpha$$
$$\phi_1'(x_0)c_1 + \phi_2'(x_0)c_2 = \beta$$

has a unique solution (c_1^*, c_2^*), by Cramer's Rule.

Consider $\psi(x) = c_1^*\phi_1(x) + c_2^*\phi_2(x)$.

Clearly $\psi(x)$ is a solution of

$$\mathcal{L}(y)=0; \quad y(x_0)=\alpha, \; y'(x_0)=\beta$$

But, by assumption $\phi(x)$ is another soln of the ODE.

Hence by uniqueness,

$$\phi(x) = \psi(x)$$
$$= c_1^*\phi_1(x) + c_2^*\phi_2(x) \quad \forall x \in I.$$

This completes the theorem.

Defn:- Any two L.I. solutions ϕ_1, ϕ_2 of $\mathcal{L}(y)=0$ form a basis or fundamental set of solutions on I.

The family of solutions $c_1\phi_1(x)+c_2\phi_2(x)$, which contains all solutions, is called the general solution of $\mathcal{L}(y)=0$.

NOTE:- Fundamental set is NOT unique for $\mathcal{L}(y)=0$.

EXM: $\{1, x\}$ and $\{2, 2x\}$ both are basis to $y''=0$.

Non-homogeneous ODEs :-

$$\mathcal{L}(y) := y'' + p(x)\, y' + q(x)\, y = f(x).$$

i.e. $\quad \mathcal{L}(y) = f(x). \quad \dots\dots (11).$

Problems :-

1. Show that $y_1 = e^t$, $y_2 = e^{-2t}$ are fundamental solutions to $y'' + y' - 2y = 0$.

2. Show that, $y_1(t) = \sin t$ & $y_2(t) = t\sin(t)$ are L.I.

3. Find the Wronskian of ~~the~~ two solutions of the eqn $t^2 y'' - (t+2)t\, y' + (t+2)\, y = 0$, $t > 0$, given that, $W(1) = e$.

   $\left[\text{Ans: } W(t) = t^2 e^t \right]$

— Any solution of (11) of the form $y = y_p$ is called a particular solution. Let, S_p be the set of all solutions of (11).

Lemma 4. Let, ϕ_1, ϕ_2 be two L.I solutions of $\mathcal{L}(y) = 0$. Then for the equation $\mathcal{L}(y) = f(x)$, the following are equivalent :

a) $y \in S_p$

b) $y = c_1 \phi_1(x) + c_2 \phi_2(x) + y_p$; for some $y_p \in S_p$.

Proof :- (b) $\Rightarrow$ (a)

$$\mathcal{L}(y) = \mathcal{L}(c_1\phi_1 + c_2\phi_2) + \mathcal{L}(y_p)$$
$$= 0 + f(x) = f(x).$$

So, $y \in S_p$.

(a) $\Rightarrow$ (b). Let, $y \in S_p$ and $y_p \in S_p$ be a particular solution.

Then

$$\mathcal{L}(y-y_p) = \mathcal{L}(y) - \mathcal{L}(y_p)$$
$$= 0$$

So, by Theorem 12, $y - y_p$ has the form:

$$y - y_p = c_1\phi_1 + c_2\phi_2$$

$$\Rightarrow \underline{y = c_1\phi_1 + c_2\phi_2 + y_p}.$$

$\Rightarrow$ The general solution of $\mathcal{L}(y) = 0$ is known as complementary solution, denoted by y_c.

That is,
$$y_c(x) = c_1\phi_1(x) + c_2\phi_2(x).$$

Therefore, the general solution of $\mathcal{L}(y) = f$ may be written as:
$$y(x) = y_c(x) + y_p(x).$$

<u>Theorem 13</u> (Method of Variation of Parameters)

A particular solution of $y'' + p(x)y' + q(x)y = f(x)$ ...(11) is given by

$$y_p(x) = \int^x \frac{\phi_1(t)\phi_2(x) - \phi_1(x)\phi_2(t)}{W(\phi_1, \phi_2; t)} f(t)\, dt$$

where $\{\phi_1, \phi_2\}$ is a fundamental set of $\mathcal{L}(y) = 0$ in I.

<u>Proof</u>:- We are looking for solution of (11) of the form

$$y_p(x) = c_1(x)\,\phi_1(x) + c_2(x)\,\phi_2(x).$$

where $c_1(x)$ and $c_2(x)$ are two unknown function.

Since y_p is a solution to (11),

we have,

$$2(c_1'\phi_1' + c_2'\phi_2') + c_1''\phi_1 + c_2''\phi_2 + p(x)\left[c_1'\phi_1 + c_2'\phi_2\right]$$
$$+ c_1(\phi_1'' + p(x)\phi_1' + q(x)\phi_1) + c_2(\phi_2'' + p(x)\phi_2' + q(x)\phi_2)$$
$$= f(x).$$

Since $\phi_1(x)$ & $\phi_2(x)$ are solutions to $L(y)=0$, we have further

$$2(c_1'\phi_1' + c_2'\phi_2') + c_1''\phi_1 + c_2''\phi_2 + p(x)\{c_1'\phi_1 + c_2'\phi_2\} = f(x). \quad \ldots (III)$$

To find $c_1(x)$ and $c_2(x)$, we need two equations in c_1 & c_2:

Consider the first equation as:

$$c_1'\phi_1 + c_2'\phi_2 = 0 \quad \ldots (IV), \quad \forall x \in I.$$

So, (III) $\Rightarrow$

$$2(c_1'\phi_1' + c_2'\phi_2') + c_1''\phi_1 + c_2''\phi_2 = f(x).$$

Now from (IV) $\Rightarrow$ $c_1'\phi_1' + c_2'\phi_2' = -c_1''\phi_1 - c_2''\phi_2$

Therefore

$$c_1'\phi_1' + c_2'\phi_2' = f(x) \quad \ldots (V).$$

So, (IV) & (V) are two equations in c_1 & c_2. (or c_1', c_2')

Now, since $W(\phi_1, \phi_2; x) \neq 0$, (IV) & (V) are solvable in c_1', c_2'.

By Cramer's Rule,

$$c_1'(x) = -\frac{\phi_2(x) f(x)}{W(\phi_1,\phi_2;x)}, \quad c_2'(x) = \frac{\phi_1(x) f(x)}{W(\phi_1,\phi_2;x)}$$

Therefore,
$$c_1(x) = -\int^x \frac{\phi_2(t) f(t)}{W(\phi_1,\phi_2;t)}\,dt, \quad c_2(x) = \int^x \frac{\phi_1(t) f(t)}{W(\phi_1,\phi_2;t)}\,dt.$$

Thus,
$$y_p(x) = \int^x \frac{\phi_1(t)\phi_2(x) - \phi_1(x)\phi_2(t)}{W(\phi_1,\phi_2;t)} f(t)\,dt.$$

EXM :- $y'' + y = \tan x$.

Note that a fundamental solution is $\{\cos x, \sin x\}$.

Now, $W(\phi_1, \phi_2; x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$.

$$y_p(x) = C_1(x)\cos x + C_2(x)\sin x.$$

$$y_p(x) = \int^x \frac{\cos t \sin x - \cos x \sin t}{1} \cdot \tan t \, dt$$

$$= \sin x \int^x \cos t \tan t \, dt - \cos x \int^x \tan t \sin t \, dt$$

$$= -\sin x \cos x - \cos x \int^x \sin^2 t \sec t \, dt$$

$$= -\sin x \cos x - \cos x \int^x \sec t \, dt + \cos x \int^x \cos t \, dt$$

$$= -\sin x \cancel{\cos x} + \cancel{\cos x \sin x} - \cos x \ln|\sec x + \tan x|$$

$$= -\cos x \ln|\sec x + \tan x|$$

Therefore, the general solution is:

$$y = C_1 \cos x + C_2 \sin x - \cos x \ln|\sec x + \tan x|.$$

H.W.

1. Solve : $y'' - y = e^x$ [Ans: $y_p(x) = xe^x/2 - e^x/4$.]

2. $y'' + y = \sec x$ [Ans: $y_p(x) = x\sin x + \cos x \ln|\cos x|$]

3. $ty'' - (1+t)y' + y = t^2 e^{2t}$ [Ans: $y_p(t) = \frac{t+1}{2} e^{2t}$.

4. $y'' = x^3$

5. $y'' + y' = \ln|x|$

6. $y'' - 2y' + y = e^x$.