Power Series Method :-

Analiticity :-

A function $f : (a,b) \to \mathbb{R}$ is said to be real analytic at $x_0 \in (a,b)$, if it can be represented by a Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(x_0)}{n!} (x - x_0)^n,$$

in a neighbourhood of x_0.

Exm: $\sin x$, $\cos x$, e^x.

Consider the homogeneous differential equation:

$$y'' + p(x)y' + q(x)y = 0. \quad \cdots \quad (1)$$

Defn :- If at a point $x = x_0$, $p(x)$ and $q(x)$ are analytic, then x_0 is called an ordinary or regular point of the ODE (1). Else, x_0 is called a singular point of the ODE.

Note :

$$f(x) = \begin{cases} \exp(-1/x^2), & x > 0 \\ 0, & x \le 0 \end{cases}$$

Check $f'(0) = f''(0) = f^{(3)}(0) = 0 = \cdots = f^{(n)}(0)$. $\forall n$.

But f is not analytic at $x = 0$.

Hence, C^{∞} functions may not be analytic.

Theorem 14 :- Let, $x = x_0$ be an ordinary point of (1). Then every solution of (1) is analytic at $x = x_0$ and can be written in the form: $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ with $R > 0$. radius of convergence.

Legendre Equation :-

$$(1-x^2)\, y'' - 2x y' + \alpha(\alpha+1)\, y = 0, \qquad \alpha \in \mathbb{R}.$$

We have,

$$y'' - \frac{2x}{1-x^2}\, y' + \frac{\alpha(\alpha+1)}{(1-x^2)}\, y = 0.$$

Therefore, $p(x) = -\frac{2x}{1-x^2}$, $q(x) = \frac{\alpha(\alpha+1)}{(1-x^2)}$ are analytic at $x=0$. So, $x=0$ is an ordinary pt. of the Legendre equation.

Let us assume one solution of the L.E as:

$$y(x) = \sum_{n=0}^{\infty} a_n (x-0)^n.$$

$$\therefore\ y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Therefore, we have

$$\sum_{n=0}^{\infty}\Big[(n+2)(n+1)\, a_{n+2} - (n-1)n a_n - 2n a_n + \alpha(\alpha+1) a_n\Big] x^n = 0$$

Hence,

$$a_{n+2} = -\frac{(\alpha-n)(\alpha+n+1)}{(n+1)(n+2)}\, a_n, \qquad n = 0, 1, 2, \ldots$$

This relation determines $a_2, a_4, a_6, \ldots$ in terms of a_0 and $a_3, a_5, a_7, \ldots$ in terms of a_1.

We write,

$$a_2 = -\frac{\alpha(\alpha+1)}{2!}\, a_0, \quad a_4 = +\frac{(\alpha-2)(\alpha+3)(\alpha+1)\alpha}{4!}\, a_0$$

$$\ldots$$

and

$$a_3 = -\frac{(\alpha-1)(\alpha+2)}{3!}\, a_1, \quad a_5 = +\frac{(\alpha-3)(\alpha-4)(\alpha-1)(\alpha+2)}{5!}\, a_1$$

So in general, $k \geq 1$,

$$a_{2k} = (-1)^k \frac{(\alpha-2k+2) \cdots (\alpha-2)\alpha(\alpha+1)(\alpha+3)\cdots(\alpha+2k-1)}{(2k)!}\, a_0$$

and $a_{2k+1} = (-1)^k \dfrac{(\alpha-2k+1)\cdots(\alpha-3)(\alpha-1)(\alpha+2)\cdots(\alpha+2k)}{(2k+1)!} a_1$

Hence, the solution of the Legendre eqn is :

$$y(x) = a_0 \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k \alpha(\alpha-2)\cdots(\alpha-2k+2)(\alpha+1)(\alpha+3)\cdots(\alpha+2k-1)}{(2k)!} x^{2k} \right]$$

$$+ a_1 \left[x + \sum_{k=1}^{\infty} \frac{(-1)^k (\alpha-1)(\alpha-3)\cdots(\alpha-2k+1)(\alpha+2)(\alpha+4)\cdots(\alpha+2k)}{(2k+1)!} x^{2k+1} \right]$$

$$= a_0 \Phi_\alpha(x) + a_1 \Psi_\alpha(x).$$

It can be seen that, Φ_α & Ψ_α are L.I and convergent for $|x|<1$.

To see L.I, just check,

$$W(\Phi_\alpha, \Psi_\alpha; 0) = \begin{vmatrix} \Phi_\alpha(0) & \Psi_\alpha(0) \\ \Phi'_\alpha(0) & \Psi'_\alpha(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

Legendre Polynomial :-

If $\alpha = m$, a non-negative integer, then one of $\Phi_m(x)$ or $\Psi_m(x)$ become a polynomial, according to m is even or odd.

Moreover, for any non-negative integer m; either $\Phi_m(x)$ or $\Psi_m(x)$, but not both, is a polynomial of degree m. We denote the polynomial as $P_m(x)$. a_0 and a_1 are choosen such that $P_m(1) = 1$. $\forall m$.

The general form of Legendre polynomial of degree m is :

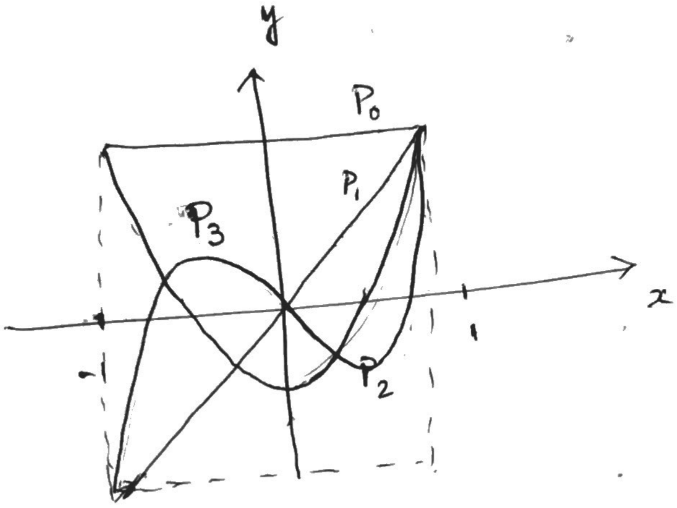
$$P_m(x) = \sum_{k=0}^{N} (-1)^k \frac{(2m-2k)!}{2^m k!\,(m-k)!\,(m-2k)!} x^{m-2k}.$$

where $N = \left[\frac{m}{2}\right] = \begin{cases} m/2, & m \text{ even} \\ (m-1)/2, & m \text{ odd}. \end{cases}$

The first few Legendre polynomials are :

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \; \ldots\ldots$$



__NOTE :-__ For non-negative m, only one of $\Phi_m(x)$ or $\Psi_m(x)$ of Legendre equation is a polynomial, while the other is an infinite series. That infinite series, with appropriately normalized, is called the Legendre function of the second kind. It is defined for $|x| < 1$ by :

$$Q_m(x) = \begin{cases} \Phi_m(1)\,\Psi_m(x), & m \text{ even} \\ -\Psi_m(1)\,\Phi_m(x), & m \text{ odd}. \end{cases}$$

So, the general solution of the Legendre eqn, for non-neg. integer m, is :

$$y(x) = c_1 P_m(x) + c_2 Q_m(x).$$

__Properties :-__

1. $P_m'(1) = \frac{1}{2}m(m+1)$

2. $P_m'(-1) = (-1)^{m-1}\frac{1}{2}m(m+1)$

3. $\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0, \quad m \neq n$ [Orthogonality]

4. $\int_{-1}^{1} (1-t^2)\,P_n'\,P_m'\,dt = 0; \quad m \neq n$

1. $P_m(x)$ satisfies the Legendre eq^n:

$$(1-x^2)y'' - 2xy' + m(m+1)y = 0.$$

So, $$(1-x^2)P_m'' - 2xP_m' + m(m+1)P_m = 0.$$

At $x=1$, $$-2P_m'(1) + m(m+1)P_m(1) = 0$$

$$\therefore \quad P_m'(1) = \frac{1}{2}m(m+1) \qquad \left[\text{By def}^n: \; P_m(1) = 1\right].$$

2. We have
$$(1-x^2)P_m''(x) - 2xP_m'(x) + m(m+1)P_m(x) = 0$$

At $x=-1$, $$2P_m'(-1) + m(m+1)P_m(-1) = 0$$

$$\Rightarrow \quad P_m'(-1) = -\frac{1}{2}m(m+1)P_m(-1).$$

$$P_m(-1) = \sum_{k=0}^{N}(-1)^k \frac{(2m-2k)!}{2^m k!(m-k)!(m-2k)!}(-1)^{m-2k}.$$

$$= (-1)^m \sum_{k=0}^{N}(-1)^k \frac{(2m-2k)!}{2^m k!(m-k)!(m-2k)!}$$

$$= (-1)^m P_m(1) = (-1)^m.$$

So, $$P_m'(-1) = (-1)^{m-1}\frac{1}{2}m(m+1).$$

3. We have,
$$\frac{d}{dx}\left[(1-x^2)P_n'\right] = -n(n+1)P_n \quad \cdots (I)$$

$$\& \quad \frac{d}{dx}\left[(1-x^2)P_m'\right] = -m(m+1)P_m \quad \cdots (II)$$

$(I) \times P_m - (II) \times P_n \Rightarrow$

$$\left[m(m+1) - n(n+1)\right]P_nP_m = P_m\frac{d}{dx}\left[(1-x^2)P_n'\right] + P_n\frac{d}{dx}\left[(1-x^2)P_m'\right]$$

Integrating over -1 to 1,

$$[m(m+1)-n(n+1)]\int_{-1}^{1} P_n P_m\,dx = (1-x^2)(P_n' P_m - P_m' P_n)\Big]_{-1}^{1}$$
$$- \int_{-1}^{1} (P_m' P_n' - P_n' P_m')(1-x^2)\,dx$$
$$= 0.$$

As $m \neq n$, $\underline{\int_{-1}^{1} P_n P_m\,dx = 0}$.

4. $$\int_{-1}^{1} (1-t^2) P_n' P_m'\,dt = \left[(1-t^2) P_n' P_m\right]_{-1}^{1} - \int_{-1}^{1} \left[(1-t^2) P_n'' - 2t P_n'\right] P_m\,dt$$
$$= + \int_{-1}^{1} n(n+1) P_n P_m\,dt$$
$$= 0 \quad (\text{as } n \neq m)$$

<u>H.W</u> 1. Solve $(x-1)y'' + y' + 2(x-1)y = 0$, $y(4) = 5$, $y'(4) = 0$ on $[4,\infty)$ by taking $y = \sum_{n=0}^{\infty} a_n (x-4)^n$.

2. Find the 1st four terms of the power series expansion around the pt $x = 1$ of each fundamental solns of $y'' - xy' - y = 0$.

<u>Defn</u> :- If $x = x_0$ is not an ordinary pt of the ODE $y'' + p(x)y' + q(x)y = 0$, then it is called a singular pt.

$x = x_0$ is called a regular singular pt if $(x-x_0)p(x)$ and $(x-x_0)^2 q(x)$ are analytic at $x = x_0$.

Otherwise $x = x_0$ is called irregular singular pt.

Exm :- 1. $x^2y'' - xy' + 2y = 0$

$p(x) = -\frac{1}{x}$; $q(x) = \frac{2}{x^2}$

$p(x)$ and $q(x)$ are not analytic at $x=0$. So, $x=0$ is a singular pt. But $xp(x)$ & $x^2q(x)$ are analytic at $x=0$. So, $x=0$ is a regular singular pt.

2. $y'' + \sqrt{x}\, y = 0$, $0 \le x < \infty$

$p(x) = 0$, $q(x) = \sqrt{x}$. Clearly, $p(x)$ & $q(x)$ are not analytic at $x=0$. Also, $x^2 q(x) = x^{5/2}$ is not analytic at $x=0$. So, $x=0$ is an irregular singular pt.

Thm 15 (Frobenius Method) Let, $x = x_0$ be a regular singular point of

$$y'' + p(x)y' + q(x)y = 0 \quad \cdots (*)$$

Then, $(*)$ has atleast one non-trivial solution of the form:

$$\Phi(x) = (x-x_0)^r \sum_{n=0}^{\infty} a_n (x-x_0)^n.$$

Where, r is called the characteristic exponent, a constant, real or complex, which is obtained by assuming $a_0 \neq 0$. The series representation is valid in $0 < |x-x_0| < R$.

NOTE :- r is a root of $r(r-1) + p_0 r + q_0 = 0$ where, $p(x_0) = p_0$ and $q(x_0) = q_0$. The quadratic polynomial in r is called the indicial polynomial.

Theorem 16. Let, $x = x_0$ be a regular singular pt. of (*). Let, r_1, r_2 be two characteristic exponents of the eqn. such that $r_1 \geq r_2$, if they are real. Then the eqn (*) has two L.I solutions $\phi_1(x)$, $\phi_2(x)$ in $0 < |x - x_0| < R$ of the form:

(a) if $r_1 - r_2 > 0$, but not an integer, then

$$\phi_i(x) = (x-x_0)^{r_i} \sum_{n=0}^{\infty} a_n (x-x_0)^n, \quad i = 1, 2.$$

(b) if $r_1 - r_2 =$ a +ve integer, then,

$$\phi_1(x) = (x-x_0)^{r_1} \sum_{n=0}^{\infty} a_n (x-x_0)^n, \quad a_0 \neq 0$$

$$\phi_2(x) = (x-x_0)^{r_2} \sum_{n=0}^{\infty} b_n (x-x_0)^n + K\phi_1(x) \ln(x-x_0).$$

with $b_0 \neq 0$ and K may or may not be zero.

(c) if $r_1 = r_2 = r$, then

$$\phi_1(x) = (x-x_0)^{r} \sum_{n=0}^{\infty} a_n (x-x_0)^n, \quad a_0 \neq 0$$

$$\phi_2(x) = (x-x_0)^{r+1} \sum_{n=0}^{\infty} b_n (x-x_0)^n + \phi_1(x) \ln(x-x_0).$$

NOTE :-

1. A series of the form $\sum_{n=0}^{\infty} a_n (x-x_0)^{n+r}$ is called a quasi-power series, $a_0 \neq 0$.

2. Solutions of the indicial polynomial may be complex as well. But we will not consider those cases, as we are looking for only real solutions.