A. To Find y_p:

1) Variation of parameters (ϕ_1, ϕ_2 known)

2. Operator method $\left[\frac{1}{D+2} e^x = \frac{1}{3} e^x. \right.$

3. Undetermined co-efficient. [Right hand side is in special form]

i) $y'' + y = 32x^3$.

Let $y_p(x) = C_1 x^3 + C_2 x^2 + C_3 x + C_4$

ii) $y'' - 2y' - 3y = 6e^{-x} - 8e^x$

$y_c = Ae^{-x} + Be^{3x}$.

Let, $y_p(x) = C_1 x e^{-x} + C_2 e^x$.

iii) $y'' + 9y = \cos 3x$

$y_p(x) = x(C_1 \cos 3x + C_2 \sin 3x)$

B. One of ϕ_1, ϕ_2 is known (Reduction of order)

$t y'' - (1+t) y' + y = 0$, $\phi_1(t) = e^t$.

Let, $y = v e^t$.

Then, $t v'' + (t-1) v' = 0$ [First order in v']

$\Rightarrow v' = k t e^{-t} \Rightarrow v = -k(t+1) e^{-t}$

So, $y = -k(t+1)$.

Take $\underline{k=1}$.

So, $\phi_2(t) = -(t+1)$

C. Cauchy Euler eqn:

$$2x^2 y'' + 3xy' - 3y = x^3.$$

Let, $x = e^t$. $\Rightarrow t = \ln x$. $\Rightarrow \frac{dt}{dx} = \frac{1}{x}$.

So, $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt}$.

$\Rightarrow D' \equiv xD$. Ily. $D'(D'-1) \equiv x^2 D^2$.

So, $\left[2D'(D'-1)+3D'-3\right]y = e^{3t}$.

$\Rightarrow (2D'^2+D'-3)\,y = e^{3t}$.

$y_c(t) = Ae^{t} + Be^{-3/2\,t} = Ax + Bx^{-3/2}$

$$y_p(t) = \frac{e^{3t}}{(2D'^2+D'-3)} = \frac{e^{3t}}{18+3-3} = \frac{e^{3t}}{18}.$$

So, $y = Ax + Bx^{-3/2} + \dfrac{x^3}{18}$

D. Finding one ϕ_1 by trial:

$y''+Py'+Qy=0$.

i) If $m(m-1)+Pmx+Qx^2=0$, then $y=x^m$ is one soln.

ii) If $m^2+mP+Q=0$, then $y=e^{mx}$ is one soln.

Try: $y'' - \dfrac{2}{x^2}y = 0$ $\qquad \phi_1 = x^2,\ \phi_2 = \dfrac{1}{x}$.

E. Reduction to Normal form: (Dependent Variable change)

– $y''+Py'+Qy=R$

is reduced to $\dfrac{d^2v}{dx^2} + Iv = S$ by changing

$y = uv$, with $u = e^{-\frac{1}{2}\int P\,dx}$, $I = Q - \dfrac{P^2}{4} - \dfrac{1}{2}\dfrac{dP}{dx}$

& $S = \dfrac{R}{u}$.

Exm:- $y'' - 2\tan x \cdot y' + 5y = \sec x \cdot e^x$.

$P = -2\tan x$, $Q = 5$.

$u = e^{+\frac{1}{\cancel{2}}\int \tan x\,dx} = e^{\ln \sec x} = \sec x$.

and $\dfrac{d^2v}{dx^2} + \left(Q - \dfrac{P^2}{4} - \dfrac{1}{2}\dfrac{dP}{dx}\right)v = e^x$.

$\Rightarrow \dfrac{d^2v}{dx^2} + 6v = e^x \quad \Rightarrow v(x) = c_1\cos(x\sqrt{6}) + c_2\sin(x\sqrt{6}) + \dfrac{e^x}{7}$

F Changing Independent Variable.

$$y'' + Py' + Qy = R$$

is changed to:

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$$

with $P_1 = \dfrac{z'' + Pz'}{(z')^2}$, $Q_1 = \dfrac{Q}{(z')^2}$, $R_1 = \dfrac{R}{(z')^2}$

Now choose Q_1 & P_1 as constants if possible.

Exm :- $(1+x^2)^2 y'' + 2x(1+x^2)y' + 4y = 0$

$$P = \frac{2x}{1+x^2}, \quad Q = \frac{4}{(1+x^2)^2}, \quad R = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = 1, \text{ let,}$$

then, $\left(\dfrac{dz}{dx}\right)^2 = \dfrac{4}{(1+x^2)^2}$

$$\Rightarrow \frac{dz}{dx} = \frac{2}{1+x^2} \Rightarrow z = 2\tan^{-1}x.$$

Then,
$$P_1 = \frac{\dfrac{-4x}{(1+x^2)^2} + \dfrac{2x}{(1+x^2)} \cdot \dfrac{2}{(1+x^2)}}{\left(\frac{dz}{dx}\right)^2} = 0.$$

& $R_1 = 0$.

So,
$$\frac{d^2y}{dz^2} + y = 0.$$

$$\Rightarrow y(z) = A\sin z + B\cos z$$

$$\therefore y(x) = A\sin(2\tan^{-1}x) + B\cos(2\tan^{-1}x).$$

$$= B\frac{1-x^2}{1+x^2} + A\frac{2x}{1+x^2}.$$