Bessel's equation :-

The Bessel's eqn is :

$$x^2 y'' + x y' + (x^2 - \alpha^2) y = 0, \quad x > 0, \quad \cdots (*)$$

where, $\alpha \geq 0$.

Since, $x p(x) = 1$, $x^2 q(x) = (x^2 - \alpha^2)$, (Regular Singular pt) it has a non-trivial soln of the form:

$$y(x) = x^r \sum_{n=0}^{\infty} a_n (x-0)^n, \quad a_0 \neq 0.$$

From $(*)$, we have,

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} - \alpha^2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

So, $a_0 [r(r-1) + r - \alpha^2] = 0 \Rightarrow r = \pm \alpha.$ $[\because a_0 \neq 0]$

Case-I. $r = \alpha$.

Consider the co-eff. of x^{r+1},

$$[(r+1)^2 - \alpha^2] a_1 = 0$$

$$\Rightarrow (1 + 2\alpha) a_1 = 0 \Rightarrow a_1 = 0 \quad [\because \alpha \geq 0]$$

Now, Consider the co-eff. of x^{n+r}, $n \geq 2$.

$$[(n+\alpha)(n+\alpha-1) + (n+\alpha)] a_n + a_{n-2} - \alpha^2 a_n = 0$$

$$\therefore a_n = - \frac{a_{n-2}}{n(n+2\alpha)}, \quad n \geq 2.$$

So, $a_1 = a_3 = a_5 = \cdots = 0.$

and

$$a_{2m} = (-1)^m \frac{a_0}{2^{2m} m! (1+\alpha)(2+\alpha) \cdots (m+\alpha)}, \quad m = 1, 2, \ldots$$

$$= (-1)^m \cdot \frac{a_0 2^{\alpha} \cdot \Gamma(\alpha+1)}{2^{2m+\alpha} m! \Gamma(\alpha+m+1)}.$$

Hence the solution becomes:

$$y_1(x) = a_0 \sum_{m=0}^{\infty} \frac{(-1)^m 2^{\alpha} \Gamma(\alpha+1)}{2^{2m+\alpha} m! \Gamma(\alpha+m+1)} x^{2m+\alpha}.$$

We choose $a_0 = \frac{1}{2^{\alpha}\Gamma(\alpha+1)}$ and denote the corresponding solution as $J_\alpha(x)$.

$$J_\alpha(x) := \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+\alpha}}{2^{2m+\alpha} m! \Gamma(\alpha+m+1)} \quad \cdots (1)$$

This is called the <u>Bessel function of 1st kind of order α.</u>

Now, $r_1 - r_2 = 2\alpha$ and let, 2α is not an integer.

<u>Case - II</u> $r = -\alpha$.

$$[(n-\alpha)(n-\alpha-1) + (n-\alpha) - \alpha^2] a_n + a_{n-2} = 0, \quad n \geq 2.$$

$$\Rightarrow n(n-2\alpha) a_n = - a_{n-2}$$

$$\therefore a_n = - \frac{a_{n-2}}{n(n-2\alpha)}, \quad n \geq 2 \quad [2\alpha \text{ is not an integer}]$$

And $(1-2\alpha) a_1 = 0 \Rightarrow a_1 = 0 \quad [\because 2\alpha \neq 1]$

So, $a_1 = a_3 = \cdots = 0$

and $$a_{2m} = (-1)^m \cdot \frac{2^{-\alpha} \Gamma(1-\alpha)}{2^{2m-\alpha} m! \Gamma(-\alpha+m+1)} a_0$$

We get the soln

$$y_2(x) = a_0 \sum_{m=0}^{\infty} (-1)^m \frac{2^{-\alpha}\Gamma(1-\alpha)}{2^{2m-\alpha} m! \Gamma(m-\alpha+1)} x^{2m-\alpha}$$

Let, $a_0 = \frac{1}{2^{-\alpha}\Gamma(1-\alpha)}$ and then

$$y_2(x) = J_{-\alpha}(x).$$

So, the general solution of $(*)$ is

$$y(x) = C_1 J_\alpha(x) + C_2 J_{-\alpha}(x)\,, \quad \text{where, } 2\alpha \text{ is not an integer.}$$

Here, $J_\alpha(x)$ & $J_{-\alpha}(x)$ converges for all $x > 0$ and are L.I.

Note :-

1. $J_\alpha(0) = 0$ and $J_{-\alpha}(0)$ diverges for $\alpha \geq 0$.

So, the solution of Bessel eqn corresponding to $r = \alpha$ is called regular soln. and the one for $r = -\alpha$ is called irregular solution.

2. From 1, $J_\alpha(x)$ cannot be proportional to $J_{-\alpha}(x)$. So, they are L.I.

3. 2α is integer, when $\alpha =$ integer or $\alpha = k + \frac{1}{2}$.

But $J_{-(k+\frac{1}{2})}(x)$ is well-defined [See it by choosing $a_0 = 1$]

4. $\Gamma(z)$ is defined by:

$$\Gamma(z) := \frac{\Gamma(z+N)}{z(z+1)\cdots(z+N-1)}\,, \quad -N < \operatorname{Re} z \leq -N+1$$

$\operatorname{Re} z < 0$ & z is not a negative integer

5. $\Gamma(m-\alpha+1)$ is defined for $m = 0, 1, 2, \ldots$, provided α is not a +ve integer.

6. If $\alpha =$ +ve integer, then by Theorem 16, we have 2nd solution as:

$$y_2(x) = x^{-N} \sum_{n=0}^{\infty} b_n x^n + c \ln x . J_N(x) \quad \ldots (A)$$

with $b_0 \neq 0$

This leads to a Bessel func. of order N of 2nd kind.

For $\alpha = n$ integer,

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+n)!} \left(\frac{x}{2}\right)^{2m+n}, \quad x \geq 0. \quad \ldots (11)$$

Properties :-

1. $J_{-n}(x) = (-1)^n J_n(x)$, $n = +ve$ integer

2. $J_{-n}(x) = (-1)^n J_n(x)$, n any integer

3. $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$, $x > 0$

4. $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$, $x > 0$

Proof :-

1. $$J_{-n}(x) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!\,\Gamma(m-n+1)} \left(\frac{x}{2}\right)^{2m+n}$$

Now, for $m = 0, 1, \ldots, (n-1)$, $\Gamma(m-n+1)$ becomes infinity, so the terms vanishes. Thus,

$$J_{-n}(x) = \sum_{m=n}^{\infty} (-1)^m \frac{1}{m!\,(m-n)!} \left(\frac{x}{2}\right)^{2m+n}$$

Let, $k = m-n$, then

$$J_{-n}(x) = \sum_{k=0}^{\infty} (-1)^{k+n} \frac{1}{(k+n)!\,k!} \left(\frac{x}{2}\right)^{2k+n}$$

$$= (-1)^n J_n(x).$$

2. Let, $n = -p$, a neg. integer

We have, $J_{-p}(x) = (-1)^p J_p(x)$.

$\therefore$ $J_p(x) = (-1)^{-p} J_{-p}(x)$

i.e. $J_{-n}(x) = (-1)^n J_n(x)$

3. $$J_\alpha(x) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}$$

$$= \left(\frac{x}{2}\right)^\alpha \frac{1}{\Gamma(\alpha+1)} \left[1 - \frac{x^2}{2^2 \cdot (\alpha+1)} + \frac{x^4}{2^4 \cdot 2!\,(\alpha+1)(\alpha+2)} - \cdots\right]$$

Now, $$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right]$$ $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

$$= \sqrt{\frac{2}{\pi x}} \cos x .$$

ll.ly, $$J_{1/2}(x) = \sqrt{\frac{x}{2}} \frac{2}{\sqrt{\pi}} \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$= \sqrt{\frac{2}{\pi x}} \sin x .$$

5. $$\lim_{t\to 0} \frac{J_\alpha(t)}{t^\alpha} = \frac{1}{2^\alpha \Gamma(\alpha+1)}, \quad \underline{\alpha > -1} .$$

By defn,
$$J_\alpha(t) = \frac{t^\alpha}{2^\alpha \cdot \Gamma(\alpha+1)} \left[1 - \frac{t^2}{4(\alpha+1)} + \frac{t^4}{4.8.(\alpha+1)(\alpha+2)} - \cdots\right]$$

So, $$\lim_{t\to 0} \frac{J_\alpha(t)}{t^\alpha} = \lim_{t\to 0} \frac{1}{2^\alpha \Gamma(\alpha+1)} \left[1 - \frac{t^2}{4(\alpha+1)} + \frac{t^4}{4.8.(\alpha+1)(\alpha+2)} - \cdots\right]$$

$$= \frac{1}{2^\alpha \Gamma(\alpha+1)} .$$

NOTE :- For n integers, $J_n(x)$ and $J_{-n}(x)$ are not L.I.
Hence, $y(x) = c_1 J_n(x) + c_2 J_{-n}(x)$ is not a general solution of Bessel equation, when n is an integer.

Thm :- Two independent solutions of Bessel's eqn may be taken as $J_\alpha(x)$ and
$$Y_\alpha(x) = \frac{\cos(\alpha\pi)\, J_\alpha(x) - J_{-\alpha}(x)}{\sin(\alpha\pi)}, \text{ for all } \alpha.$$

So, the general solution of Bessel's eqn, when, n is an integer and $x>0$ is:

$$y(x) = c_1 J_n(x) + c_2 Y_n(x).$$

$Y_n(x)$ is called the Bessel's fn of 2nd kind of order n.

This $Y_n(x)$ ∴ and ($\ast$) are of same form.

Hermite Equation :-

$$x'' - 2tx' + 2\alpha x = 0$$

Clearly, $p(t) = -2t$, $q(t) = 2\alpha$. Both are analytic at $t=0$.

So, the soln is of the form:

$$x(t) = \sum_{n=0}^{\infty} a_n t^n.$$

So,

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} - 2\sum_{n=1}^{\infty} n a_n t^n + 2\alpha \sum_{n=0}^{\infty} a_n t^n = 0$$

We have,

$$2a_2 + 2\alpha a_0 = 0 \Rightarrow a_2 = -\alpha a_0$$

$$3.2.a_3 - 2a_1 + 2\alpha a_1 = 0, \Rightarrow a_3 = \frac{2(1-\alpha)}{3.2.} a_1$$

In gen,

$$a_{2m} = -\frac{2^m \alpha(2-\alpha)(4-\alpha)\dots(2m-2-\alpha)}{(2m)!} a_0, \quad m = 2,3,\dots$$

$$\& \ a_{2m+1} = \frac{2^m (1-\alpha)(3-\alpha)\dots(2m-1-\alpha)}{(2m+1)!} a_1, \quad m = 1,2,\dots$$

So, the soln is:

$$x(t) = a_0 \left[1 - \alpha t^2 - \sum_{m=2}^{\infty} \frac{2^m \alpha(2-\alpha)\dots(2m-2-\alpha)}{(2m)!} t^{2m}\right]$$

$$+ a_1 \left[t + \sum_{m=1}^{\infty} \frac{2^m (1-\alpha)(3-\alpha)\dots(2m-1-\alpha)}{(2m+1)!} t^{2m+1}\right]$$

$$= a_0 \Phi_\alpha + a_1 \Psi_\alpha$$

Since, $W(\Phi_\alpha, \Psi_\alpha; 0) = 1 \neq 0$, they are L.I.

If $\alpha = K$, a non-neg. integer, then one of Φ_α or Ψ_α become a polynomial, formally called Hermite polynomial of order K, $H_K(t)$, with suitable a_0, a_1.

Rodrigues' Formula :-

$$H_K(t) = (-1)^K e^{t^2} \frac{d^K}{dt^K}\left(e^{-t^2}\right), \quad K = 0, 1, 2, \ldots$$

$$H_0(t) = 1, \; H_1(t) = 2t, \; H_2(t) = 4t^2 - 2, \; H_3(t) = 8t^3 - 12t, \ldots$$

In general,
$$H_K(t) = K! \sum_{n=0}^{[\frac{K}{2}]} \frac{(-1)^n}{n!\,(K-2n)!} (2t)^{K-2n}, \quad t \in \mathbb{R}.$$

Chebyshev Equation :-

$$(1-t^2)x'' - tx' + \alpha^2 x = 0$$

$t = 0$ is an ordinary point. So, the soln can be taken as:
$$x(t) = \sum_{n=0}^{\infty} a_n t^n.$$

So,
$$(1-t^2)\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} - \sum_{n=1}^{\infty} n a_n t^n + \alpha^2 \sum_{n=0}^{\infty} a_n t^n = 0$$

Hence,
$$2a_2 + \alpha^2 a_0 = 0 \implies a_2 = -\frac{\alpha^2}{2} a_0$$
$$3.2\, a_3 - a_1 + \alpha^2 a_1 = 0 \implies a_3 = \frac{(1-\alpha^2)}{3!} a_1.$$

In general,
$$(n+2)(n+1)a_{n+2} - n(n-1)a_n - na_n + \alpha^2 a_n = 0$$
$$\implies a_{n+2} = \frac{(n^2-\alpha^2)}{(n+2)(n+1)} a_n, \quad n = 0, 1, 2, \ldots$$

Therefore,
$$a_{2m} = -\frac{((2m-2)^2-\alpha^2)\cdots(4^2-\alpha^2)(2^2-\alpha^2)\alpha^2}{(2m)!} a_0, \quad m = 2, 3, \ldots$$
$$a_{2m+1} = \frac{((2m-1)^2-\alpha^2)\cdots(3^2-\alpha^2)(1^2-\alpha^2)}{(2m+1)!} a_1, \quad m = 1, 2, \ldots$$

So, the solution is:

$$x(t) = a_0 \left[1 - \frac{\alpha^2}{2} t^2 - \sum_{m=2}^{\infty} a_{2m} t^{2m} \right]$$

$$+ a_1 \left[t + \sum_{m=1}^{\infty} a_{2m+1} t^{2m+1} \right]$$

$$= a_0 \Phi_\alpha(t) + a_1 \Psi_\alpha(t).$$

Φ_α & Ψ_α are L.I and conv. for $|t|<1$.

If $\alpha = K$, a non-neg. integer, Φ_K or Ψ_K becomes a polynomial, formally known as Chebyshev polynomial, (after properly normalized)

$$(1-t^2)x'' - tx' + K^2 x = 0 \quad \cdots (1)$$

Let, $t = \cos z$.

$$\frac{dx}{dz} = \frac{dx}{dt} \cdot \frac{dt}{dz} = -\sin z \cdot \frac{dx}{dt}$$

$$\frac{d^2x}{dz^2} = -\cos z \cdot \frac{dx}{dt} + \sin^2 z \cdot \frac{d^2x}{dt^2}.$$

So, (1) $\Rightarrow$ $\frac{d^2x}{dz^2} + K^2 x = 0$

$\therefore$ $x(z) = A\cos(Kz) + B\sin(Kz)$

$\therefore$ $x(t) = A\cos(K\cos^{-1}t) + B\sin(K\cos^{-1}(t)).$

$= A\,T_K(t) + B\,U_K(t).$ $\quad |t|<1$

Defn :-

Chebyshev polynomial of first kind is defined by:

$$T_n(t) = \cos(nz) \quad \text{where, } t = \cos z.$$

$T_0(t) = 1$, $T_1(t) = t$, $T_2(t) = 2t^2 - 1$, ...

Airy Equation :-

$$y'' - xy = 0$$

$x = 0$ is an ordinary pt. So, we take the eqn as:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

So, $a_2 = 0$, $a_3 = \dfrac{a_0}{3.2}$, $a_4 = \dfrac{a_1}{4.3}$.

& $a_{n+2} = \dfrac{a_{n-1}}{(n+2)(n+1)}$

In general, $a_{3k} = \dfrac{(3k-2)(3k-5)\cdots 4.1}{(3k)!} a_0$, $k = 1, 2, \ldots$

$$a_{3k+1} = \frac{(3k-1)(3k-4)\cdots 2}{(3k+1)!} a_1, \quad k = 1, 2, \ldots$$

$$a_{3k+2} = 0, \quad k = 0, 1, 2, \ldots$$

So, the soln is:

$$y(x) = a_0 \left[1 + \sum_{k=1}^{\infty} a_{3k} x^{3k}\right] + a_1 \left[\cancel{t}\, x + \sum_{k=1}^{\infty} a_{3k+1} x^{3k+1}\right]$$

$$= a_0 \Phi(x) + a_1 \Psi(x).$$

Φ, Ψ are L.I and conv. for all $x \in \mathbb{R}$.

Problems :-

1. Prove that the Bessel function $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}} \sin x$ satisfies the Bessel eqn of order $1/2$.

2. Solve $t^2 x'' - t x' - (t^2 + 5/4) x = 0$ using Frobenius Method.

3. Solve: $2x^2 y'' + (x^2 - x) y' + y = 0$

   $x^2 y'' + (x - x^2) y' + y = 0$

4. Solve: $xy'' + (1-x) y' + \alpha y = 0$ (Laguerre eqn)

5. Given $y_1(x) = x$ is a soln. of $x^3 y'' + x y' - y = 0$, find a 2nd independent soln $y_2(x)$. [Use $y_2 = v \cdot y_1$]

Show that $y_2(x)$ is not a Frobenius series.

6. Find the zeros of Chebyshev polynomial $T_n(x)$ of degree n. [$x_k = \cos\left(\frac{(2k-1)\pi}{2n}\right)$, $k = 1, 2, \ldots, n$]