Boundary Value Problems & Strum-Liouville Theory

BCM ODEs

Defn [Two point BVP]

The mathematical problem of solving the 2nd order equation

$$L(x) = x''(t) + p(t)\,x'(t) + q(t)\,x(t) = -f(t) \quad \text{(I)}$$

together with boundary condns

$$\left.\begin{aligned} B_1[x, t_1] &= b_{11}x(t_1) + b_{12}x'(t_1) = \alpha \\ B_2[x, t_2] &= b_{21}x(t_2) + b_{22}x'(t_2) = \beta \end{aligned}\right\} \quad \text{(II)}$$

is called a BVP., where b_{ij}, α, β are real constants.

NOTE :- In IVP, if $p(t)$, $q(t)$, $f(t)$ are continuous, then irrespective to the initial condition, the IVP has a unique solution. But this is NOT true for BVP.

Exm. $y'' + y = 1$, $y(0) = 0$, $y(\pi) = 0$

The gen. soln is $y(x) = c_1 \cos x + c_2 \sin x + 1$.

Now,
$$y(0) = c_1 + 1 = 0$$
$$y(\pi) = -c_1 + 1 = 0$$

which is impossible. So the BVP has no solution.

Exm. $y'' + y = 0$

I) $y(0) = 1$, $y(\pi) = 0$ II) $y(0) = 1$, $y(\pi/2) = 1$ III) $y(0) = 1$, $y(\pi) = -1$

I) No solution II) Unique solution III) Infinite solution.

Theorem 17. Let, p, q, f are cont. on $[t_1, t_2]$. Then either the BVP (I)-(II) has a unique solution for any α, β or else the associated homogeneous BVP

$$\left.\begin{array}{c} \mathcal{L}(x) = 0 \\ B_1[x, t_1] = 0, \quad B_2[x, t_2] = 0 \end{array}\right\} \cdots \text{(III)}$$

has a non-trivial solution.

Proof :- Let, ϕ_1 & ϕ_2 be solutions of

$$\mathcal{L}(\phi_1) = 0, \quad \phi_1(t_1) = b_{12}, \quad \phi_1'(t_1) = -b_{11}$$

$$\mathcal{L}(\phi_2) = 0, \quad \phi_2(t_2) = b_{22}, \quad \phi_2'(t_2) = -b_{21}$$

and $\psi(t)$ be a solution of

$$\mathcal{L}(\psi) = -f(t), \quad \psi(t_1) = 0 = \psi'(t_1)$$

Since, p, q, f are cont., the IVPs have unique solutions.

Define $\quad \phi(t) = c_1\phi_1(t) + c_2\phi_2(t) + \psi(t)$

Clearly, $\mathcal{L}(\phi) = -f(t)$. We will check under what conditions, the Boundary conditions (II) are satisfied.

$$\alpha = B_1[\phi, t_1] = c_1 B_1[\phi_1, t_1] + c_2 B_1[\phi_2, t_1] + B_1[\psi, t_1]$$

$$= c_1[b_{11}\phi_1(t_1) + b_{12}\phi_1'(t_1)] + c_2[b_{11}\phi_2(t_1) + b_{12}\phi_2'(t_1)] + [b_{11}\psi(t_1) + b_{12}\psi'(t_1)]$$

$$\Rightarrow c_2[\phi_1(t_1)\phi_2'(t_1) - \phi_2(t_1)\phi_1'(t_1)] = \alpha \quad \cdots (*)$$

Ill.ly, $\quad \beta = B_2[\phi, t_2]$

$$\Rightarrow -c_1[\phi_1(t_2)\phi_2'(t_2) - \phi_1'(t_2)\phi_2(t_2)] + B_2[\psi, t_2] = \beta. \quad \cdots (**)$$

Case - I $\quad \phi_1, \phi_2$ are L.I.

$(*) - (**) \Rightarrow$

$$c_2 W(\phi_1, \phi_2; t_1) = \alpha$$
$$-c_1 W(\phi_1, \phi_2; t_2) + B_2[\psi, t_2] = \beta.$$

So, c_1 & c_2 are uniquely determined. So, the BVP (I)-(II) has unique solution.

To prove that, the homogeneous BVP has only trivial solution.

Since, ϕ_1, ϕ_2 are L.I, any solution of $L(x)=0$ can be written as

$$x(t) = k_1\phi_1(t) + k_2\phi_2(t).$$

If it satisfies $B_1[x, t_1] = 0$, $B_2[x, t_2] = 0$, then

$$b_{11} k_1 \phi_1(t_1) + b_{11} k_2 \phi_2(t_1) + b_{12} k_1 \phi_1'(t_1) + k_2 b_{12} \phi_2'(t_1) = 0$$

$$\Rightarrow k_2 W(\phi_1, \phi_2; t_1) = 0 \Rightarrow k_2 = 0$$

||ly, $\quad k_1 = 0$

So, the solution is trivial.

Case - II $\quad \phi_1, \phi_2$ are L.D.

Then, c_1, c_2 are not uniquely determined from $(*) - (**)$. So, the BVP can not have unique solution.

Also, k_1 and k_2 can take non-zero values. So, the homogeneous BVP has non-trivial solution.

Exm :- $\quad x'' + x = 1$

$x(0) = 0 = x(\pi)$

The soln of homogeneous BVP is

$$x(t) = c \sin t.$$

i.e. non-trivial soln.

$x'' + x = 1$

$x(0) = 0 = x(\pi/2)$

The BVP has unique soln:

$$x(t) = 1 - \cos t - \sin t.$$

And the homogeneous BVP has only trivial soln.

Adjoint Forms :-

Consider the linear equation:

$$\mathcal{L}(x) = a_0(t)\, x'' + a_1(t)\, x' + a_2(t)\, x = 0 \quad , \; t \in I. \quad \cdots (i)$$

Integrate $z\mathcal{L}(x)$ over a to t, $a \in I$.

$$\int_a^t z\mathcal{L}(x)\, dt = \int_a^t \left[(z a_0 x'' + z a_1 x') + z a_2 x \right] dt$$

$$= \left[z a_0 x' + z a_1 x \right]_a^t - \int_a^t \left[(z a_0)' x' + (z a_1)' x - z a_2 x \right] dt$$

Now, $$\int_a^t (z a_0)' x'\, dt = \left[(z a_0)' x \right]_a^t - \int_a^t (z a_0)'' x\, dt$$

So, $$\int_a^t z\mathcal{L}(x)\, dt = \left[(z a_0) x' - (z a_0)' x + z a_1 x \right]_a^t + \int_a^t \left[(z a_0)'' - (z a_1)' + z a_2 \right] x\, dt$$

Define the operator

$$\mathcal{L}^*(z) := (z a_0)'' - (z a_1)' + (z a_2)$$

$$= a_0 z'' + (2a_0' - a_1) z' + (a_0'' - a_1' + a_2) z \quad \cdots (ii)$$

Then, we have

$$\int_a^t \left[z\mathcal{L}(x) - x\mathcal{L}^*(z) \right] dt = \left[a_0(x'z - xz') + (a_1 - a_0') xz \right]_a^t \quad \cdots (iii)$$

The operator $\mathcal{L}^*$ is called the adjoint operator corresponding to $\mathcal{L}$.

$$\mathcal{L}^*(z) = a_0 z'' + (2a_0' - a_1) z' + (a_0'' - a_1' + a_2) z$$
$$= b_0 z'' + b_1 z' + b_2 z \text{, let.}$$

Now, $(\mathcal{L}^*)^*(x) = b_0 x'' + (2b_0' - b_1) x' + (b_0'' - b_1' + b_2) x$

$$= a_0 x'' + (2a_0' - 2a_0' + a_1) x' + (a_0'' - 2a_0'' + a_1' + a_0'' - a_1' + a_2) x$$

$$= a_0 x'' + a_1 x' + a_2 x$$

$$= \mathcal{L}(x).$$

So, $(\mathcal{L}^*)^*(x) = \mathcal{L}(x)$.

Defn :- The 2nd order homogeneous linear equation $\mathcal{L}(x) = 0$ is called self-adjoint if

$$\mathcal{L}^*(x) \equiv \mathcal{L}(x).$$

NOTE: Then

$$\left.\begin{aligned} a_1 &= 2a_0' - a_1 \\ a_2 &= a_0'' - a_1' + a_2 \end{aligned}\right\} \Rightarrow a_1 = a_0'$$

Result :- Let $a_0 \in C^2$, $a_1 \in C^1$ and $a_2 \in C$ with $a_0(t) \neq 0$ on I. A neccessary and sufficient condition that $\mathcal{L}(x) = 0$ be self-adjoint is that

$$a_0' = a_1, \text{ on } I.$$

If $\mathcal{L}(x) = 0$ is self-adjoint, then

$$\mathcal{L}(x) = \frac{d}{dt}\left(a_0 \frac{dx}{dt}\right) + a_2 x = 0.$$

NOTE :- A general operator $\mathcal{L}$ may not be self adjoint but it can be converted into a self-adjoint by multiplying a suitable function $h(t)$.

Lemma 5 The operator $h(t)\mathcal{L}(x)$ is self adjoint,

where
$$h(t) = \frac{1}{a_0(t)} \exp\left\{ \int^t \frac{a_1(s)}{a_0(s)}\, ds \right\}.$$

In fact $h(t)\mathcal{L}(x) = 0$ can be written as:

$$\left[p(t)\, x'(t) \right]' + q(t)\, x(t) = 0$$

with
$$p(t) = \exp\left\{ \int^t \frac{a_1(s)}{a_0(s)}\, ds \right\}$$
$$q(t) = \frac{a_2(t)}{a_0(t)} \exp\left\{ \int^t \frac{a_1(s)}{a_0(s)}\, ds \right\}$$

EXM: Legendre : $(1-t^2)x'' - 2tx' + \alpha(\alpha+1)x = 0$

$$\Rightarrow \frac{d}{dt}\left[(1-t^2)x' \right] + \alpha(\alpha+1)x = 0$$

Bessel: $x^2 y'' + xy' + (x^2 - \alpha^2)y = 0$

$$\Rightarrow \frac{d}{dx}\left[xy' \right] + \left(x - \frac{\alpha^2}{x} \right) y = 0$$

From (iii) $\Rightarrow$
$$\int_a^t \left[z\mathcal{L}(x) - x\mathcal{L}^*(z) \right] dt = \left[a_0(x'z - z'x) + (a_1 - a_0')xz \right]_a^t$$

Differentiating both sides w.r.to t.
$$z\mathcal{L}(x) - x\mathcal{L}^*(z) = \frac{d}{dt}\left[a_0(x'z - z'x) + (a_1 - a_0')xz \right]$$

This is called <u>Lagrange Identity</u>.

If $t = b$, then, we get

$$\int_a^b [z\mathcal{L}(x) - x\mathcal{L}^*(z)]\,dt = \left[a_0(x'z - z'x) + (a_1 - a_0')xz\right]_a^b$$

This is called <u>Green's Identity</u>.

If $\mathcal{L}$ is self-adjoint, then

Green's Identity:

$$\int_a^b [z\mathcal{L}(x) - x\mathcal{L}(z)]\,dt = \left[a_0(x'z - xz')\right]_a^b$$

Lagrange Identity:

$$z\mathcal{L}(x) - x\mathcal{L}(z) = \frac{d}{dt}\left[a_0(x'z - xz')\right]$$

<u>EXM</u> :- 1. $xy'' + 2y' + \alpha^2 xy = 0$, $x > 0$

$$h(x) = \frac{1}{x}\exp\left(\int^x \frac{2}{x}\,dx\right)$$

$$= x.$$

So, the self-adjoint form is:

$$(x^2y')' + \alpha^2 x^2 y_1 = 0. \checkmark$$

2. $t^2x'' - 2tx' + 2x = 0$

$$h(t) = \frac{1}{t^2}e^{-\int \frac{2t}{t^2}dt}$$

$$= \frac{1}{t^2}\cdot e^{-\ln t^2} = \frac{1}{t^4}.$$

Multiplying with $\frac{1}{t^4}$:

$$\frac{1}{t^2}x'' - \frac{2}{t^3}x' + \frac{2}{t^4}x = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{t^2}x'\right) + \frac{2}{t^4}x = 0.$$

Defn :- (Strum - Liouville Problem)

The BVP consisting of

i) a second order homogeneous linear ODE

$$\frac{d}{dx}\left[p(x)\, y'\right] + \left[q(x) + \lambda r(x)\right] y = 0 \quad (*)$$

where p, q, r are real valued fn., with q, r cont. and p continuously differentiable, and $p(x), r(x) > 0$ in $[a,b]$ and λ be a parameter.

ii) two BCs

$$\left.\begin{aligned} b_{11}\, y(a) + b_{12}\, y'(a) &= 0 \\ b_{21}\, y(b) + b_{22}\, y'(b) &= 0 \end{aligned}\right\} \cdots (**)$$

is called a SLP problem. (regular).

EXM :- 1. $(1-t^2)x'' - 2t x' + (n+1)n x = 0$

$$\frac{d}{dt}\left[(1-t^2)x'\right] + n(n+1)x = 0$$

2. $t^2 x'' + t x' + (t^2 - \alpha^2)x = 0$

$$\frac{d}{dt}(t x') + \left(t - \frac{\alpha^2}{t}\right)x = 0$$

Defn :- A non-trivial soln $y(x)$ of $(*)-(**)$, if it exists, is called an eigen-function and the corresponding value of λ is called the eigen-value of $(*)-(**)$.

NOTE :- A BVP is called a singular SLP if

i) $p > 0$ on (a,b) & $p(a) = p(b) = 0$ and

ii) $r \geq 0$ on $[a,b]$.

EXM :- $y'' + \lambda y = 0$, $y(0) = 0 = y'(\pi)$. , $\lambda \in \mathbb{R}$.

For $\lambda < 0$ or $\lambda = 0$, $y(x) \equiv 0$ is the only solution. So, there are no negative or zero eigenvalues.

Let, $\lambda = \mu^2 > 0$.

Then, the general sol^n is

$$y(x) = A\cos(\mu x) + B\sin(\mu x).$$

$$\Rightarrow A = 0 \quad \text{and} \quad B\cos(\mu\pi) = 0$$

So, for non-trivial sol^n $\cos(\mu\pi) = 0$

$$\Rightarrow \mu\pi = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow \mu_n = \frac{(2n+1)}{2}, \quad n = 0, 1, 2 \ldots$$

Therefore, the eigenvalues are

$$\lambda_n = \frac{(2n+1)^2}{4}; \quad n = 0, 1, 2, \ldots$$

and the corresponding eigen functions are:

$$y_n(x) = B\sin\left(\frac{2n+1}{2}x\right), \quad n = 0, 1, 2, \ldots$$

EXM. $y'' + \lambda y = 0$, $y(0) - y(\pi) = 0$, $y'(0) - y'(\pi) = 0$

$\lambda < 0$ yields trivial sol^n.

$\lambda = 0 \Rightarrow y(x) = A + Bx$.

So, $B = 0$ and A is arbitrary.

So, 0 is an eigen value with eigen function being any non-zero constant.

$\lambda = \mu^2 > 0$.

Then, $y(x) = A\cos(\mu x) + B\sin(\mu x)$

So, $A\sin(\mu\pi) + B(1-\cos(\mu\pi)) = 0$

$A(1-\cos(\mu\pi)) - B\sin\mu\pi = 0$

This has non-trivial soln if

$$\begin{vmatrix} \sin\mu\pi & 1-\cos(\mu\pi) \\ 1-\cos\mu\pi & -\sin\mu\pi \end{vmatrix} = 0$$

i.e. $\cos(\mu\pi) = 1 \Rightarrow \mu = 2n, \ n = 1, 2, \dots$

$\Rightarrow \lambda_n = 4n^2$

and the corresponding eigenfunctions are:

$y_n(x) = \cos(2nx), \ z_n(x) = \sin(2nx). \ n \in \mathbb{N}.$

Therefore, there are two L.I. eigenfunctions corresponding to each $\lambda_n = 4n^2$.

<u>H.W</u>. $x^2y'' + xy' + \lambda y = 0, \ 1 \le x \le e, \ y(1) = 0 = y(e)$.

$\frac{d}{dx}(xy') + \frac{1}{x}\lambda y = 0$

<u>H.W</u> $y'' + \lambda y = 0, \ -\pi \le x \le \pi, \ y(-\pi) = y(\pi)$

$y'(-\pi) = y'(\pi)$

For $\lambda = \mu^2 > 0$.

$y(x) = A\sin(\mu x) + B\cos(\mu x)$

So, $2A\sin(\mu\pi) = 0$ & $2\mu B\sin(\mu\pi) = 0$

For non-trivial soln, $\sin(\mu\pi) = 0, A \neq 0, B \neq 0$

$\therefore \mu_n = n, \ n \in \mathbb{N}$.

So, the eigen-fn are

$y_n(x) = \sin(nx), \ z_n(x) = \cos(nx)$

For $\lambda = 0$, the eigenfunction is: $y_n(x) = 1$

For $\lambda < 0$, there are no eigen functions.

Theorem 18 : Let, the co-eff. p, q, r in SLP be cont. on $[a, b]$. Let, ϕ_1 & ϕ_2 be two eigen functions corresponding to λ_1 & λ_2. Then ϕ_1 & ϕ_2 are orthogonal w.r.to the weight function r in $[a, b]$.

Proof :- Since ϕ_i, correspondig to λ_i satisfies the Strum-Liouville equation, we have

$$\frac{d}{dx}(p\phi_i') + (q + \lambda_i r)\phi_i = 0. \quad , i = 1, 2.$$

Then, $$(\lambda_1 - \lambda_2) r\phi_1\phi_2 = \phi_2 \frac{d}{dx}(p\phi_1') - \phi_1 \frac{d}{dx}(p\phi_2')$$
$$= \frac{d}{dx}\left[(p\phi_1')\phi_2 - (p\phi_2')\phi_1\right]$$

and by integration,

$$(\lambda_1 - \lambda_2)\int_a^b r\phi_1\phi_2\,dx = p(b)\left[\phi_1'(b)\phi_2(b) - \phi_2'(b)\phi_1(b)\right] - p(a)\left[\phi_1'(a)\phi_2(a) - \phi_2'(a)\phi_1(a)\right]$$

The BCs of eigen functions are:

$$B_2[\phi_1, b] = 0, \quad B_2[\phi_2, b] = 0$$

$$\Rightarrow \quad b_{21}\phi_1(b) + b_{22}\phi_1'(b) = 0$$
$$\& \quad b_{21}\phi_2(b) + b_{22}\phi_2'(b) = 0$$

If $b_{22} \neq 0$, then, $b_{22}\left[\phi_1'(b)\phi_2(b) - \phi_1(b)\phi_2'(b)\right] = 0$

$$\Rightarrow \quad \phi_1'(b)\phi_2(b) - \phi_1(b)\phi_2'(b) = 0$$

Similarly, if $b_{12} \neq 0$, we have

$$\phi_1'(a)\phi_2(a) - \phi_2'(a)\phi_1(a) = 0.$$

Hence, $$(\lambda_1 - \lambda_2)\int_a^b r\phi_1\phi_2\,dx = 0$$

If $\lambda_1 \neq \lambda_2$, then $\displaystyle\int_a^b r\phi_1\phi_2\,dx = 0$.

Theorem 19

All the eigen-values of a regular SLP are real.

Proof :- Suppose there is a complex eigen-value $\lambda_1 = \alpha + i\beta$ with eigen function $\phi_1 = u + iv$. Since the co-eff of SLP ~~are~~ are real, the complex conjugate of λ_1 is also an eigen-value.

So, $\exists$ an eigen function $\phi_2 = u - iv$ corresponding to the eigen value $\lambda_2 = \alpha - i\beta$.

Then, we have

$$(\lambda_1 - \lambda_2) \int_a^b r \phi_1 \phi_2 \, dx = 0$$

$$\Rightarrow 2i\beta \int_a^b r(u^2+v^2) \, dx = 0$$

$$\Rightarrow 2\beta \int_a^b r(u^2+v^2) \, dx = 0$$

$$\Rightarrow \beta = 0$$

So, all the eigen-values are real.

Lemma 6

If ϕ_1 & ϕ_2 are two solns of $L(x) + \lambda r(t) x = 0$ on $[a,b]$, then $p(t) \, W(\phi_1, \phi_2; t) =$ Constant.

Proof :- We have

$$\frac{d}{dt}\left(p(t)\frac{d\phi_1}{dt}\right) + (q + \lambda r)\phi_1 = 0$$

$$\& \quad \frac{d}{dt}\left(p\frac{d\phi_2}{dt}\right) + (q + \lambda r)\phi_2 = 0$$

Eliminating q, λ, r we get,

$$\phi_1 \frac{d}{dt}(p\phi_2') - \phi_2 \frac{d}{dt}(p\phi_1') = 0$$

Integrating we get,

$$\int_a^t \frac{d}{dt}\left(p\phi_1\phi_2' - p\phi_2\phi_1'\right)\,dt = 0$$

$$\Rightarrow \quad p(t)\left[\phi_1(t)\phi_2'(t) - \phi_2(t)\phi_1'(t)\right] = p(a)\left[\phi_1(a)\phi_2'(a) - \phi_1'(a)\phi_2(a)\right]$$

$$\Rightarrow \quad \underline{p(t)\,W(\phi_1,\phi_2;t) = \text{Constant}}.$$

<u>Lemma 7</u> An eigen function of a regular SLP is unique upto a constant factor.

<u>Proof :-</u>

Let, ϕ_1 & ϕ_2 are two eigen functions corresponding to the eigen-value λ.

Then, by Lemma 6,

$$p(t)\,W(\phi_1,\phi_2;t) = K, \quad p(t) > 0.$$

So, if $W(\phi_1,\phi_2;t)$ vanishes at one point in $[a,b]$, it will vanish for all $t \in [a,b]$.

Now, ϕ_1, ϕ_2 satisfy:

$$B_1[\phi_1,a] = b_{11}\phi_1(a) + b_{12}\phi_1'(a) = 0$$

$$B_1[\phi_2,a] = b_{11}\phi_2(a) + b_{12}\phi_2'(a) = 0$$

Since, $(b_{11}, b_{12}) \neq (0,0)$, so $W(\phi_1,\phi_2;a) = 0$.

Therefore, $W(\phi_1,\phi_2;t) = 0 \quad \forall\, t \in [a,b]$.

i.e. ϕ_1, ϕ_2 are L.D. $\Rightarrow \phi_2 = c\phi_1$ for some constant c.

Theorem 20. (Existence of Eigen functions)

The regular SLP $(*)-(**)$ has an infinite sequence of real eigen-values $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ with

$$\lim_{n\to\infty} \lambda_n = \infty.$$

The eigenfunction ϕ_n corresponding to λ_n has exactly n zeros in (a,b). Moreover, ϕ_n is unique upto a constant factor.

Proof:- For proof, see Coddington & Levinson / Tyn-Mint-U.

Exm:- $y'' + \lambda y = 0, \quad y(0) = 0 = y'(\pi)$

$$\lambda_n = \frac{(2n+1)^2}{4}, \quad n = 0,1,2,\ldots$$

$$y_n(x) = B_n \sin\left(\frac{2n+1}{2}x\right).$$

Fact:

1. $\frac{1}{4} = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$

   & $\lim_{n\to\infty} \lambda_n = \infty$

2. $\phi_n(x) = \sin\left(\frac{2n+1}{2}x\right)$ is an eigen fn corresponding to $\lambda_n = \left(\frac{2n+1}{2}\right)^2$.

   $\phi_0(x) = \sin\left(\frac{x}{2}\right)$ has no zeros in $(0,\pi)$

   $\phi_1(x) = \sin\left(\frac{3x}{2}\right)$ has only one zero at $\frac{2\pi}{3}$ in $(0,\pi)$

   $\vdots$

   $\phi_n(x) = \sin\left(\frac{2n+1}{2}x\right)$ has n zeros at

   $$x = \frac{2m\pi}{(2n+1)}, \quad m = 1,2,\ldots,n \text{ in } (0,\pi)$$

Problems :-

1. Let, $h > 0$ be a real no. Find eigen-values and corresponding eigen-vectors of the SLP:

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) + h y'(1) = 0.$$

2. Verify theorem 20 for

$$\frac{d}{dx}[x y'] + \frac{\lambda}{x} y = 0$$

$$y'(1) = 0, \quad y'(e^{2\pi}) = 0$$

3. Verify theorem 20 for

$$y'' + \lambda y = 0, \quad y(0) = 0 = y(\pi).$$

4. Expand the function $f(x) = x$ using eigen functions of

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(1) = 0$$

i.e. $$f(x) = \sum_{n=1}^{\infty} c_n y_n(x).$$

5. Solve: $$\frac{d}{dx}(e^x y') + \lambda e^x y = 0, \quad y(0) = 0, \quad y(1) = 0$$ for non-trivial solutions.