Separation & Comparison Theory

Consider the 2nd order linear homogeneous eqn:

$$(p(x)y)' + q(x)y = 0 , \quad x \in [a,b] \quad \cdots \text{ (1)}$$

Let, $p \in C^1$ and $q \in C$, $p > 0$ on $[a,b]$.

Lemma 8. Let, ϕ be a solution of (1). If ϕ has an infinite no. of zeros on $[a,b]$, then

$$\phi(t) \equiv 0 , \quad t \in [a,b]$$

Proof :- Since there are infinite no of zeros of ϕ in $[a,b]$, by Bolzano-Weierstrass, $\exists$ a seqn $\{x_n\}$ of zeros of ϕ which converges to $x_0 \in [a,b]$.

i.e. $\lim_{n\to\infty} x_n = x_0$.

Since ϕ is continuous on $[a,b]$,

$$\phi(x_0) = \lim_{n\to\infty} \phi(x_n) = 0.$$

Also, $$\phi'(x_0) = \lim_{x\to x_0} \frac{\phi(x) - \phi(x_0)}{x - x_0} \quad \text{exists.}$$

$$= \lim_{x\to x_0} \frac{\phi(x)}{x - x_0}$$

Now if $x \to x_0$ through the seqn $\{x_n\}_n$, then

$$\phi'(x_0) = 0.$$

Thus ϕ is a soln of (1) with $\phi(x_0) = 0 = \phi'(x_0)$.

Then, by uniqueness, $\phi(x) \equiv 0$ on $[a,b]$.

Lemma 9. If ϕ_1 & ϕ_2 be any two solutions of ①, then

$$p(x)\, W(\phi_1, \phi_2; x) = K, \text{ constant } \forall x \in [a,b].$$

Lemma 10. Let, ϕ_1, ϕ_2 are two solutions of ①, having a common zero $x_0 \in [a,b]$. Then, ϕ_1, ϕ_2 are L.D.

Proof :-

We have,

$$p(x)\, W(\phi_1, \phi_2; x) = K.$$

Since, $\phi_1(x_0) = \phi_2(x_0) = 0$, $W(\phi_1, \phi_2; x_0) = 0$

Thus, $K = 0$. Since, $p(x) > 0$ in $[a,b]$,

$$W(\phi_1, \phi_2; x) = 0 \quad \forall x.$$

So, ϕ_1, ϕ_2 are L.D.

Lemma 11. Let, ϕ_1, ϕ_2 be two non-trivial L.D. solutions of ① with $\phi_1(x_0) = 0$. Then $\phi_2(x_0) = 0$.

Proof :-

Since ϕ_1, ϕ_2 are L.D, $\phi_1 = c\phi_2$ $\forall x \in [a,b]$, $c \neq 0$.

Since $\phi_1(x_0) = 0$, we get

$$\phi_2(x_0) = 0.$$

Theorem 21 (Sturm Separation Theorem)

If ϕ_1 and ϕ_2 are L.I. solutions of

$$(p(x)y')' + q(x)y = 0 \quad \text{on } [a,b], \text{ then}$$

between any two successive zeros of ϕ_1, $\exists$ exactly one zero of ϕ_2, i.e. zeros of ϕ_1 & ϕ_2 occurs alternatively.

Proof :- Let, x_1, x_2 be two consecutive zeros of ϕ_1 in $[a,b]$.

By Lemma 10, $\phi_2(x_1) \neq 0$, $\phi_2(x_2) \neq 0$.

If possible, let, ϕ_2 has no zero in (x_1, x_2).

Define the function $\phi : [x_1, x_2] \to \mathbb{R}$ by

$$\phi(x) = \frac{\phi_1(x)}{\phi_2(x)}, \quad x \in [x_1, x_2]$$

Now, $\phi(x_1) = 0 = \phi(x_2)$, ϕ is diff. in $[x_1, x_2]$.

So, by Rolle's thm,

$$\phi'(x_0) = 0$$

$$\Rightarrow \frac{W(\phi_1, \phi_2; x_0)}{[\phi_2(x_0)]^2} = 0$$

$\Rightarrow W(\phi_1, \phi_2; x_0) = 0$, contradicting ϕ_1, ϕ_2 are L.I.

So, ϕ_2 vanishes at at least one point in (x_1, x_2).

Now, if $\exists$ two zeros of ϕ_2 in (x_1, x_2), then a similar argument shows that, $\exists$ a zero of ϕ_1 in between two zeros of ϕ_2, contradicting the fact that, x_1, x_2 are two successive zeros of ϕ_1.

This completes the proof.

Exm: 1. $y'' + y = 0$

$y_1(x) = \sin x$ & $y_2(x) = \cos x$ are L.I. solns.

zeros of y_1 & y_2 are seperated.

2. $y'' - y = 0$

$\phi_1(x) = e^x$, $\phi_2(x) = e^{-x}$ are L.I. None of them have zero.

$\psi_1(x) = \sinh(x)$, $\psi_2(x) = \cosh(x)$ are also L.I. solns.

ψ_1 has one zero, ψ_2 has none.

$\Rightarrow$ Existence of zeros of solution is guranteed by Theorem 21.

NOTE :- A general 2nd order eqn

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$ can be transformed

into Normal form :

$$u'' + q(x)u = 0$$

by changing $y = u\, e^{-\frac{1}{2}\int^x \frac{a_1(t)}{a_0(t)}dt}$

where $$q(x) = \frac{a_2}{a_0} - \frac{1}{4}\frac{a_1^2}{a_0^2} - \frac{1}{2}\frac{a_1' a_0 - a_1 a_0'}{a_0^2}$$

Theorem 22 If $q(x) \leq 0$ on I, then no non-trivial soln of $y'' + q(x)y = 0$ can have two zeros on I.

Proof :- If possible, let a non-trivial soln ϕ has two zeros on I, x_1 and x_2 with $x_1 < x_2$.

WLOG, let, $\phi(x) > 0$ in (x_1, x_2) [It can not change sign in (x_1, x_2).]

Therefore, $\phi'(x_1) > 0$ and $\phi'(x_2) < 0$

Because, if $\phi'(x_1) = 0$, then since $\phi(x_1) = 0$, $\phi(x) \equiv 0$ by uniqueness.

Also, if $\phi'(x_1) < 0$, then ϕ will be negative for $x > x_1$ and x near by x_1, contradicting $\phi > 0$ on (x_1, x_2).

So, $\phi''(x) = -q(x).\phi(x) \geq 0$ on (x_1, x_2).

Hence ϕ' is increasing on (x_1, x_2). Hence

$$0 < \phi'(x_1) \leq \phi'(x_2) < 0$$

This is a contradiction. So the non-trivial solution ϕ can not have two zeros.

Theorem 23 (Sturm Comparison Theorem)

On $[a,b]$, let $\phi_1(x)$ and $\phi_2(x)$ be solutions of

$$(p(x)y')' + Q_1(x)y = 0$$

$$\text{and } (p(x)y')' + Q_2(x)y = 0$$

respectively. Let, $p \in C^1$ and $Q_1, Q_2 \in C$ with $Q_2(x) > Q_1(x)$.

Then, in between any two successive zeros of ϕ_1 on $[a,b]$, ϕ_2 has at least one zero.

Proof :- Let, ϕ_2 does not have a zero in (x_1, x_2) with x_1 and x_2 are two consecutive zeros of ϕ_1.

W.L.O.G, let, $\phi_1(x) > 0$ and $\phi_2(x) > 0$ on (x_1, x_2).

We have

$$(p(x)\phi_1')' + Q_1\phi_1 = 0$$

$$\& \ (p(x)\phi_2')' + Q_2\phi_2 = 0 \qquad \text{for } x \in [a,b].$$

Now we get,

$$\phi_2 \frac{d}{dx}[p\phi_1'] - \phi_1 \frac{d}{dx}[p\phi_2'] = (Q_2 - Q_1)\phi_1\phi_2$$

$$\Rightarrow \frac{d}{dx}\left[p(x)(\phi_1'\phi_2 - \phi_1\phi_2')\right] = (Q_2 - Q_1)\phi_1\phi_2$$

$$> 0 \quad \text{in } (x_1, x_2)$$

Therefore $p(x)\,(\phi_1'\phi_2 - \phi_1\phi_2')$ is montonically increasing in (x_1, x_2), in fact in $[x_1, x_2]$. ... (*)

But, $p(x_1)\,(\phi_1'\phi_2 - \phi_1\phi_2')\Big|_{x=x_1} = p(x_1)\,\phi_1'(x_1)\,\phi_2(x_1) \geq 0$

and $p(x)\,(\phi_1'\phi_2 - \phi_1\phi_2')\Big|_{x=x_2} = p(x_2)\,\phi_1'(x_2)\,\phi_2(x_2) \leq 0$

since $\phi_1'(x_1) > 0$ and $\phi_1'(x_2) < 0$.

So, a contradition arise in (*).

Hence, $\exists$ at least one zero of ϕ_2 in (x_1, x_2).

NOTE :- In thm 23, let x_1 is zero of both ϕ_1 & ϕ_2. Then if x_2 and x_3 are the next zeros of ϕ_1 & ϕ_2, then

$$x_3 < x_2.$$

EXM : $y'' + x^2 y = 0$

Consider $y'' + y = 0$. Its solution $\sin x$ has zeros at $n\pi$, $n \in \mathbb{N}$.

Since, $x^2 \geq 1$ on $[1, \infty)$, by Sturm-Comparison thm, a non-trivial solution of the given eqn has at least one zero between $n\pi$ and $(n+1)\pi$. So, it has infinitely many zeros in $[1, \infty)$.

H.W. Show that, between any two consecutive zeros of $\sin 2x + \cos 2x$ there is precisely one zero of $\sin 2x - \cos 2x$.

Defn [Oscillatory & Non-Oscillatory]

If all the non-trivial soln of $y''+q(x)y=0$ have infinite no of zeros on $[\alpha,\infty)$ for some $\alpha \in \mathbb{R}$ then the eqn is called oscillatory and the solns are called oscillatory solution.

If there is atleast one non-trivial soln which has only a finite no of zeros, then the eqn is called non-oscillatory.

EXM :- $y''+y=0$: oscillatory

$y''-y=0$: non-oscillatory.

Problems :

1. Let, $\phi(x)$ be any non-trivial soln of $L(y)=0$ in $(0,\infty)$.

a) S.t. the zeros of ϕ are isolated.

b) S.t the set of all zeros of ϕ is countable.

c) If $x_1, x_2, \ldots$ are zeros of ϕ, then S.t.

$$\lim_{n\to\infty} x_n = \infty.$$

2. If $x(t)$ is a non-trivial soln of $x''+q(t)x=0$, then show that $x(t)$ has infinite no of zeros if $q(t) > \frac{K}{4t^2}$, $K>1$ and has a finite no. of zeros if $q(t) \leq \frac{1}{4t^2}$ $\forall t \in (0,\infty)$.

3. S.t. every non-trivial solution of the eqn $x''+(\sinh t)x=0$ has at most one zero in $(-\infty,0)$ and infinitely many zeros in $(0,\infty)$.

[Hint: $\sinh t < 0$ in $(-\infty,0)$ & $\sinh t > t$ in $(0,\infty)$ & $t>1$ in $(1,\infty)$]

4. Let, $q(t) > 0$ & q is cont. on $(0,\infty)$. Prove that, every non-trivial solution of $x'' + q(t)x = 0$ has infinitely many zeros in $(0,\infty)$.

[H: $q(t) > \delta^2$ for some $\delta > 0$].

5. Show that the distance between two consecutive zeros of any solⁿ of the eqⁿ $x'' + tx = 0$ tends to zero as $t \to \infty$.

[Hint: $t > n^2$ in (n^2, ∞) for some $n \in \mathbb{N}$.]

6. S.t. $y'' - x^2 y = 0$ is non-oscillatory in $\mathbb{R}$.

7. S.t. $y'' + (1+x)y = 0$ is oscillatory in $[0,\infty)$.

8. Let, $q_1(x)$ & $q_2(x)$ be cont. and $q_1(x) \geq q_2(x)$ in I.
Show that

i) if $y'' + q_2(x) y = 0$ is oscillatory, then so is $y'' + q_1(x) y = 0$.

ii) if $y'' + q_1(x) y = 0$ is non-oscillatory, then so is $y'' + q_2(x) y = 0$.

*9. Show that every non-trivial solution of the Hermite ODE $y'' - 2xy' + 2ay = 0$ $(a \geq 0)$ has at most finitely many zeros in $\mathbb{R}$.

[Hint: $y = z e^{x^2/2}$.]

10. Show that every non-trivial solⁿ of $y'' + e^x y = 0$ has infinitely many zeros on $(0,\infty)$, whereas a non-trivial solⁿ of $y'' - e^x y = 0$ has at most one zero in $(0,\infty)$.

11. S.t. the solⁿ of $y'' + y = 0$ oscillate more rapidly than those of $x^2 y'' + xy' + y = 0$ on $(1,\infty)$.