BCM PDEs

PDE

Mathematical Foundation :-

TA : 20%, MidSem : 30%, End Sem : 50%.

Class Test 1 : 5th Feb, Class Test 2 : 9th Apr.

Books :-

1. Partial Diff. Eqn, Lawrence C. Evans.
2. Elements of partial diff. eqn, IAN N. SNEDDON.
3. Partial Diff. Eqn, an introduction, WALTER A. STRAUSS.
4. P.DEs, J. Rauch.
5. Linear PDEs for Scientists & Engineers, TYN MYINT-U & LOKENATH Debnath.

Directional Derivative :-

Let, $f : \mathbb{R}^n \to \mathbb{R}$ is a scalar fn and $\vec{d} = (d_1, d_2, \ldots, d_n)$ is a unit vector, i.e. $\|\vec{d}\| = 1$. Then the limit

$$\lim_{t \to 0} \frac{f(\vec{a} + t\vec{d}) - f(\vec{a})}{t}$$

is called the directional derivative of f at $\vec{a}$ in the direction of $\vec{d}$, and denoted by $D_{\vec{d}} f(\vec{a})$.

If f is differentiable (?) at $\vec{a}$, then,

$$D_{\vec{d}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{d}$$

Gradient (?)

Gradient :-

$f : \mathbb{R}^n \to \mathbb{R}$ is a function. The gradient of a differentiable f, ∇f, is a vector defined by

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

Jacobian :-

Let, $f: \mathbb{R}^n \to \mathbb{R}^m$ is a vector valued fn.

$f = (f_1, f_2, \ldots, f_m) \in \mathbb{R}^m$. The Jacobian of f is an $m \times n$ matrix defined by:

$$J_f = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots \frac{\partial f}{\partial x_n} \right]$$

$$= \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & & \vdots \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & & \frac{\partial f_m}{\partial x_n} \end{pmatrix}_{m \times n}$$

Derivative :-

A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is said to be differentiable at $\vec{a} \in \mathbb{R}^n$ if $\exists\ J_f : \mathbb{R}^n \to \mathbb{R}^m$ (linear map or a Jacobian)

s.t.
$$\lim_{\vec{h} \to \vec{0}} \frac{\| f(\vec{a}+\vec{h}) - f(\vec{a}) - J_f \cdot \vec{h} \|_{\mathbb{R}^m}}{\| h \|_{\mathbb{R}^n}} = 0.$$

For $m=1$, the Jacobian matrix is reduced to the gradient of f, ∇f.

Hessian matrix :-

Let, $f: \mathbb{R}^n \to \mathbb{R}$ is a fn. with its 2nd order P.ds exist and are continuous. The Hessian matrix H of f is an $n \times n$ matrix defined by

$$H_f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \vdots & & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

$$H_f = J_{\nabla f}$$

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n} \right) \equiv (f_1, \dots, f_n)$$

$$\nabla f : \mathbb{R}^n \to \mathbb{R}^n$$

$$H_f = J_{\nabla f} = \left(\frac{\partial(\nabla f)}{\partial x_1}, \frac{\partial(\nabla f)}{\partial x_2}, \cdots \frac{\partial(\nabla f)}{\partial x_n} \right).$$

Exm :-

$$f(x,y) = \begin{cases} \dfrac{3x^2y - y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & \text{else} \end{cases}$$

From defn, calculate $D_{\vec{d}} f(0,0)$ with $\vec{d} = \frac{1}{\sqrt{2}}(1,1)$.

Show that, $D_{\vec{d}} f(0,0) \neq \nabla f\big|_{(0,0)} \cdot \vec{d}$. Reason.

$$D_{\vec{d}} f(0,0) = \lim_{t\to 0} \frac{f(\vec{a}+t\vec{d}) - f(\vec{a})}{t}$$

$$= \lim_{t \to 0} \frac{(3t^3 - t^3)2}{2\sqrt{2}\,(2t^2).t} = \frac{4}{4\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{-k}{k} = -1$$

$$\therefore \nabla f\big|_{(0,0)} = (0,-1).$$

$$\therefore D_{\vec{d}} f(0,0) \neq \nabla f\big|_{(0,0)} \cdot \vec{d}.$$

Because f is not diff. Had it been diff, $J_f = \nabla f$:

& $\lim_{(h,k)\to(0,0)} \dfrac{|f(h,k) - f(0,0) - (0,-1).(h,k)|}{\sqrt{h^2+k^2}} \neq 0.$

But, $$\lim_{(h,k)\to(0,0)} \frac{|f(h,k) - f(0,0) + k|}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \left| \frac{\dfrac{3h^2k - k^3}{h^2+k^2} + k}{\sqrt{h^2+k^2}} \right|$$

$$= \lim_{(h,k)\to(0,0)} \frac{|3h^2k + h^2k|}{(h^2+k^2)^{3/2}} = \lim_{(h,k)\to(0,0)} \frac{|4h^2k|}{(h^2+k^2)^{3/2}}$$

$$\not\to 0$$

as take $y = mx$ path.

$$\lim_{(x,y)\to(0,0)} \frac{|4x^2mx|}{(1+m^2)^{3/2}x^3} = \frac{|4m|}{(1+m^2)^{3/2}}$$, dependent on m.

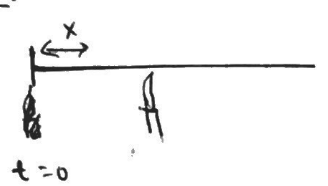
PDE :-

ODE: $f(x, y, y', y'', \ldots, y^{(n)}) = 0$

One indep. variable : x
One dep. variable : y.

Generally, physical quantities depends on both space (x) and time (t). A PDE relates the ~~variable~~ variations of the physical quantity in time & space.

Exm:



Heating of Beam

Vibration of String

Physical quantity: temperature , Displacement (vertical)

Defn :- A PDE is a relation of the form

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots) = 0$$ where

F is a given function of indep variables $x, y, \ldots$ and the unknown function u and finite no. of P.ds.

Exm: $\underline{u_x + u_t = 0}$. (Transport eqn)

Evans: $\bar{u} : \mathbb{R}^n \to \mathbb{R}^p$ (vector valued fn)

$$F(\bar{x}, \bar{u}, D\bar{u}, D^2\bar{u}, \ldots, D^k\bar{u}) = 0$$

$$D^\alpha \bar{u} = \frac{\partial^\alpha \bar{u}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \qquad \alpha := (\alpha_1, \alpha_2, \ldots, \alpha_n)$$
$$|\alpha| = \alpha_1 + \ldots + \alpha_n$$

Order: The highest derivative that appears in the equation.

Exm:- $u_x + u u_y + u_{yyy} = 0$ 3rd order.

Solution :- A function $\Phi : \mathbb{R}^n \to \mathbb{R}^p$ is said to be a solution of

$$F(\bar{x}, \bar{u}, D\bar{u}, D^2\bar{u}, \dots, D^m\bar{u}) = 0 \quad \text{if}$$

i) Φ is ~~~~ m-times differentiable.

ii) Φ satisfies:

$$F(\bar{x}, \Phi(x), D\Phi(x), D^2\Phi(x), \dots, D^m\Phi(x)) = 0$$

in some domain $U \subseteq \Omega$

\# There is no guarantee that ~~an eqn~~ a PDE will have a soln.

$$(u_x)^2 + 1 = 0 \quad : \text{ No solution.}$$

ODE: Cauchy - Peano (Continuous $\Rightarrow$ existence of soln)

$$y' = f(t, y)$$

Picard - Lindelöf (existence & Uniqueness)

There are no such specific or general existence results for PDEs!

(Algebraic Classification)

Linearity :- A PDE is said to be linear if it is linear in u & its derivatives with the co-eff. depending only on the indep. variables.

Exm: 1. $u_{tt} - c^2 u_{xx} = 0$: Wave eqn.

2. $u_t - k u_{xx} = 0$: Heat eqn.

3. $u_{xx} + u_{yy} = 0$: Laplace eq^n.

Non-linear :- A PDE, which is not linear, is called non-linear.

PDE
- Linear
- Non-linear
  - Semi-linear
  - Quasi-linear
  - Strongly Non-Linear

Grouping them for the convenience of solving each group with specific technique.

i> Semi-linear :- This non-linear PDE has only independent variables as the co-eff. of the highest order derivatives.

EXM :- $x u_x + xy^2 u_y = u^3$

ii> Quasi-linear :- A PDE of order m is called Quasi-linear if it is linear in the derivatives of order m with co-eff. depending on the indep. variables and derivatives of u of order $< m$.

EXM : a) $(1+u_x^2) u_{xx} + 2u_x u_y u_{xy} + u_x^2 u_{yy} + u = 0$

b) $\cancel{u_x + uu_y = 0}$ & $u_t + uu_x = 0$ (Burgers')

iii> Strongly non-linear :- Highest order derivatives are not linear.

EXM: $u_x^2 + u_y^2 = f(x,y)$

H.W:- Classify:

i) (Korteweg-de Vries [KdV])

$$u_t + u_{xxx} + u u_x = 0 \qquad \text{(Semi-linear)}$$

ii) $(x^2+y^2)\, u_t + u_{xy} - \cancel{3y}\, 3u = 0 \qquad (L)$

iii) $ux\, u_{xx} + u^2 xy\, u_{xy} + uy\, u_{yy} + u_x^2 + u_y^2 + u^3 = 0 \qquad (QL)$

iv) $u_{xx} + u_{xy}^2 + u_{yy} = x^2 + y^2 \qquad (NL)$

v) (Liouville eqn.) $\quad \nabla^2 u + e^{\lambda u} = 0$

Classification (Nature):-

Stationary Problem:- A PDE, not depending on time t.

Evolution Problem:- A PDE, dependis on time.

Well-Posed Problem (Hadamard)

A PDE is said to be well-posed if

i) the problem has a soln (Existence).
ii) the soln is unique (Uniqueness)
iii) the soln continuously depends on the initial data. (Stability)

i) $u'' = 0$

$u'(0) = 0$

$u'(1) = 1$

$u(x) = A + Bx$

No soln

ii) $u'' = 0$

$u'(0) = 0$.

$u'(1) = 0$

$u(x) = A$

Not unique!

iii) $u_{xx} + u_{yy} = 0, \quad y > 0$

BC: $u(x,0) = 0$

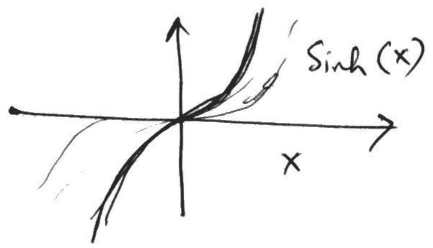
$u_y(x,0) = 0$

$u = 0$ is a soln

Now, $u_y(x,0) = e^{-\sqrt{n}} \sin(nx) \to 0$ as n

$u(x,y) = \frac{1}{n} e^{-\sqrt{n}} \sin(nx) \sinh(ny) \to \infty$ as $y \to \infty$

If a problem is well-posed, good chance of getting a sol^n with numerical techniques. If ill-posed, it has to be re-formulated (Tikhonov regularization etc.) for numerical treatment.



Differences between ODE & PDE :-

- A general sol^n of ODE involves arbitrary constants. A general sol^n of a PDE involves arbitrary functions.

i) $u_{xx} = 0$

$\Rightarrow \quad u_x = f(y)$

$\Rightarrow \quad u(x,y) = x f(y) + g(y)$

f, g : arbitrary functions.

ii) $u_{xx} + u = 0$, ODE with an extra vaiable y.

$\therefore \quad u(x,y) = f(y) \sin x + g(y) \cos x$

Homogeneous eq^n :- Each term of the PDE contains u or its derivatives.

Else, the PDE is called non-homogeneous.

$\mathcal{L}(u) = 0$ $\mathcal{L}$: Linear operator

i.e. $\mathcal{L}(u + cv) = \mathcal{L}(u) + c\mathcal{L}(v)$

$\mathcal{L}(u) = u_{tt} - c^2 u_{xx}$

Superposition principle of linear PDEs:

If u_1 & u_2 are two solutions of $\mathcal{L}(u)=0$ then, $c_1u_1+c_2u_2$ is also a sol$\underset{=}{n}$.

Non-homogeneous :-

$$\mathcal{L}(u)=f \quad \cdots (*)$$

- If u_H solves $\mathcal{L}(u)=0$ and u_p solves $(*)$ then, $u=u_H+u_p$ solves $(*)$.

- If u_1 solves $\mathcal{L}(u_1)=f_1$ and u_2 solves $\mathcal{L}(u_2)=f_2$, then, $u=c_1u_1+c_2u_2$ solves $\mathcal{L}(u)=c_1f_1+c_2f_2$.

Not true for non-linear eq$\underset{=}{n}$!

ODE $\quad yy''-xy'=0$ has sol$\underset{=}{n}$ $y_1=x^2$ & $y_2=1$

But, $y=-x^2$ is not a sol$\underset{=}{n}$ of the ODE.

PDE $\quad u_x+u^2u_y=0 \quad \cdots \boxed{*}$

$$u_1(x,y)=\frac{-1+\sqrt{1+4xy}}{2x}$$ is a sol$\underset{=}{n}$ of $\boxed{*}$. But

$u=cu_1$ is not a sol$\underset{=}{n}$ unless $c=0,\pm1$.

Solution Techniques of PDEs :-

1. Separation of variables : A PDE of n indep variables is reduced to n ODEs

2. Integral transforms : A PDE of n indep variables is reduced to a PDE of (n-1) variables (Laplace, Fourier)

3. Change of variables : A PDE can be transformed to an ODE by changing co-ordinates.

4. Change of Dependent variable : $u \rightarrow v$ (dep. variable) Easier to solve.

5. Numerical Methods : Finite Difference, Finite Element.

6. Perturbation method : A non-lin PDE is changed into a seqn of linear PDEs that approximates the original PDE.

7. Integral Equations : A PDE is changed to an integral equation.

8. Variational Method : The solution to a PDE is found by reformulating the eqn as a minimization problem. (Energy Method)

First order PDEs :-

General form:

$$F(x, y, u, u_x, u_y) = 0, \quad (x,y) \in D \subseteq \mathbb{R}^2.$$

Constant co-eff. Eqn :-

$$a u_x + b u_y = 0. \quad \cdots (*)$$

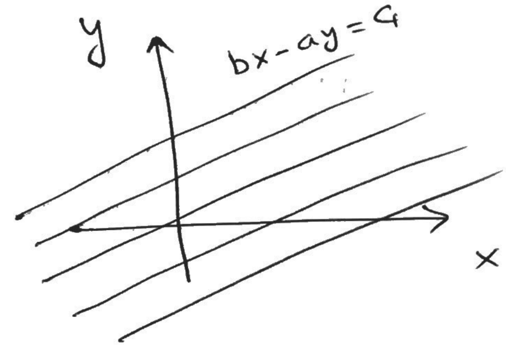
Geometric Method:

$a u_x + b u_y = (a, b) \cdot (u_x, u_y)$: directional derivative of u along (a,b). So, $(*) \Rightarrow u$ is constant along the vector (a, b).

The eqn of such lines : $bx - ay = c$.

- These are called Characteristic lines.
- $u(x,y)$ is constant on each such lines.

Let, the constant value of u along $bx - ay = c$ is $K = f(c)$



$$\therefore \ u(x,y) = f(c) = f(bx - ay)$$

Since, c is arbitary, we have $u(x,y) = f(bx - ay)$ $\forall x, y$.

Co-ordinate Method:

Change of variables:

$$x' = ax + by, \quad y' = bx - ay$$

then, $u_x = u_{x'} \dfrac{\partial x'}{\partial x} + u_{y'} \dfrac{\partial y'}{\partial x} = a u_{x'} + b u_{y'}$

& $u_y = b u_{x'} - a u_{y'}$

i.e. $(*) \Rightarrow (a^2 + b^2) u_{x'} = 0 \Rightarrow u_{x'} = 0$

$\therefore \quad u(x', y') = f(y') = f(bx - ay).$

<u>H.W:</u> Solve: $4u_x - 3u_y = 0 \; ; \; u(0, y) = y^3.$

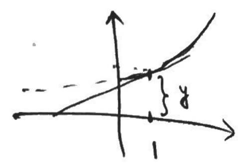
Soln: $u(x, y) = \dfrac{(3x + 4y)^3}{64}.$

Variable co-eff eqn :-

$$u_x + y u_y = 0$$

Linear, Homogeneous. The directional derivative of u along $(1, y)$ is zero. So, $u(x, y)$ is constant along those curves whose slope at (x, y) is

$$\frac{dy}{dx} = \frac{y}{1}.$$



$\Rightarrow \quad y = ce^{x}$: characteristic curves.

$\Rightarrow \quad \underline{c = ye^{-x}}.$

Illy to the previous case, u is constant along these curves and

$$\underline{u(x, y) = f(ye^{-x})}$$

<u>H.W:</u> $u_x + 2xy^2 u_y = 0$ Soln: $u(x, y) = f\left(x^2 + \frac{1}{y}\right)$

So, in general, $a(x, y)u_x + b(x, y)u_y = 0$ have characteristic eqn

$$\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)}$$

Method of Characteristic $\equiv$ PDE $\rightarrow$ ODE on characteristic curves.

Cauchy Problem for quasi-linear PDE :-

General form:

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u) \quad \cdots (*)$$

a, b, c are continuously diff. functions on $\Omega \subseteq \mathbb{R}^3$. Let, Ω_0 denotes the projection of Ω to the xy-plane.

Integral Surface :- Let, $D \subseteq \Omega_0$ and $u: D \to \mathbb{R}$ be a solⁿ of (*). The surface S represented by $z = u(x,y)$ is called an ~~ite~~ integral surface corresponding to a given solⁿ u.

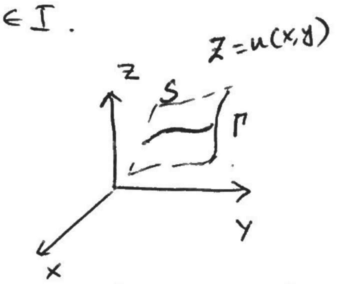
Cauchy Problem :- Cauchy problem for (*) is finding an integral surface $z = u(x,y)$ of (*) containing a given space curve Γ with parametric equations:

$$x = f(s), \quad y = g(s), \quad z = h(s), \quad s \in I$$

and

$$h(s) = u\big(f(s), g(s)\big), \quad s \in I.$$

Γ is called Data curve.



Initial Value Problem :- A special cauchy problem, where Γ lies in xz-plane, any y indicates time-variable.

$$\Gamma: \quad x = f(s), \quad y = 0, \quad z = h(s)$$

and

$$h(s) = u(f(s), 0), \quad s \in I$$

Note :-

1. Integral Surface S:

$$z = u(x,y)$$

Define $F(x,y,z) = u(x,y) - z$.

Normal at any point : (x_0, y_0, z_0) :

$\nabla F(x_0, y_0, z_0)$: gradient of F. &

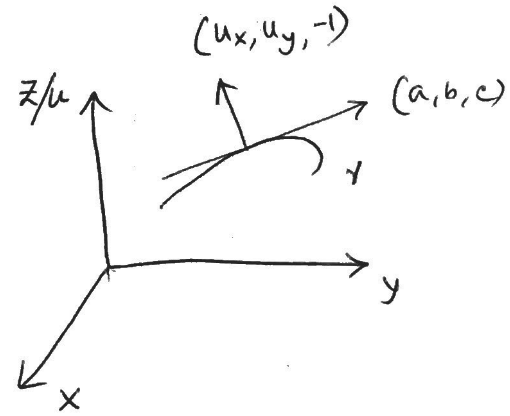
$$\nabla F = (u_x, u_y, -1) \equiv \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$$

$(*) \Rightarrow$

$$\big(a(x,y,u),\ b(x,y,u),\ c(x,y,u)\big) \cdot \nabla F = 0$$

Hence, (a,b,c) ~~belongs~~ is a tangent vector of the integral surface S. at $(x_0, y_0, u(x_0,y_0))$

This direction is called the characteristic direction or Monge axis.



2. A curve in $\mathbb{R}^3$ ~~$(x, y, u(x,y))$~~ -space whose tangent at every point coincides with the characteristic direction (a,b,c) is called a characteristic curve.

If $\gamma(t) = (x(t), y(t), z(t))$ is a characteristic curve then the tangent vector is $\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$, which is by definition:

$$(A) \ \cdots \left[\ \frac{dx}{dt} = a(x,y,u),\ \frac{dy}{dt} = b(x,y,u) \ \ \& \ \ \frac{dz}{dt} = c(x,y,u)\right.$$

$[\ z = u(x,y)$