Non-parametric form :-

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)} \quad \cdots (1)$$

Theorem :- Let, $D \subseteq \Omega_0$ and $S : z = u(x,y)$ be a surface in $\mathbb{R}^3$ where $u : D \to \mathbb{R}$ is a continuously diff. function.
Then the following are equivalent:

i) S is an integral surface of

$$a(x,y,u)\, u_x + b(x,y,u)\, u_y = c(x,y,u) \quad \cdots (\text{II})$$

ii) S is a union of characteristic ~~surf~~ curves of the PDE (II).

Proof :-

i) $\Rightarrow$ ii) Let, $S : z = u(x,y)$ is an integral surface.

So,
$$a(x,y,u)\, u_x + b(x,y,u)\, u_y = c(x,y,u), \quad (x,y) \in D.$$

To prove that, for every $P \in S$, ~~the~~ a characteristic curve γ_p through P lies entirely on S.

or to prove $S = \bigcup \gamma_p$

γ_p is a characteristic curve.

$\gamma_p : \; x = x(t), \; y = y(t), \; z = z(t), \quad t \in I.$

and $P = (x(t_0), y(t_0), z(t_0))$ for some $t_0 \in I$.

Let, $V(t) = z(t) - u(x(t), y(t))$.

As $P \in S$, $V(t_0) = 0$.

[redacted]

Now, $\dfrac{dv}{dt} = \dfrac{dz}{dt} - u_x x'(t) - u_y y'(t)$

$= c(x,y,z) - u_x a(x,y,z) - u_y b(x,y,z)$

$= c(x,y,v+u) - u_x a(x,y,v+u) - u_y \cdot b(x,y,v+u)$ ... (III)

This is an ~~ODE~~ IVP. with $v(t_0)=0$.

As a,b,c are continuously diff., we can apply Picard-Lindelöf for ODE. Moreover, $v(t)=0$ is a solⁿ of (III). So by uniqueness of solⁿ., this is the only solⁿ such that $v(t_0)=0$.

So, γ_p lies on S.

II) $\Rightarrow$ I) Let, S is a union of characteristic curves of (II).

To prove that S is an integral surface. i.e. to show $z=u(x,y)$ solves (II).

Let $P(x_0,y_0,u(x_0,y_0))$ be any point on S. Since S is a union of characteristic curves, $\exists\ \gamma_p$ through P. lies on S.

Now, the normal direction to S at P

$= (u_x(x_0,y_0), u_y(x_0,y_0), -1)$ and

$(a(x_0,y_0,u(x_0,y_0)), b(x_0,y_0,u(x_0,y_0)), c(x_0,y_0,u(x_0,y_0)))$ is the direction of tangent to γ_p. (By definition).

So, (II) is satisfied.

(Lagrange)

Theorem :- The general soln of (II) is $F(\phi, \psi) = 0$ where F is an arbitrary fn of ϕ & ψ and $\phi(x,y,u) = c_1$, $\psi(x,y,u) = c_2$ are two soln of the characteristic equations:

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c} \quad \ldots (1)$$

Proof :- Since $\phi(x,y,u) = c_1$ & $\psi(x,y,u) = c_2$, we have

$$d\phi = \phi_x\,dx + \phi_y\,dy + \phi_u\,du = 0$$

$$d\psi = \psi_x\,dx + \psi_y\,dy + \psi_u\,du = 0$$

Now, from (1), we get

$$a\phi_x + b\phi_y + c\phi_u = 0 \quad \ldots (A)$$

$$\& \quad a\psi_x + b\psi_y + c\psi_u = 0 \quad \ldots (B)$$

We now solve (A) and (B)

$$\frac{a}{\frac{\partial(\phi,\psi)}{\partial(y,u)}} = \frac{b}{\frac{\partial(\phi,\psi)}{\partial(u,x)}} = \frac{c}{\frac{\partial(\phi,\psi)}{\partial(x,y)}} \quad \ldots (*)$$

where $\frac{\partial(\phi,\psi)}{\partial(x,y)} = \begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix} \neq 0$ i.e. ϕ & ψ are l.i.

Now, we have $F(\phi, \psi) = 0$ i.e.

$$F_x = F_\phi(\phi_x + \phi_u u_x) + F_\psi(\psi_x + \psi_u . u_x) = 0$$

$$\& \quad F_y = F_\phi(\phi_y + \phi_u . u_y) + F_\psi(\psi_y + \psi_u . u_y) = 0$$

For non-trivial soln F_ϕ & F_ψ,

$$\begin{vmatrix} \phi_x + \phi_u u_x & \psi_x + u_x \psi_u \\ \phi_y + u_y \phi_u & \psi_y + \psi_u u_y \end{vmatrix} = 0$$

i.e.

$$u_x \frac{\partial(\phi,\psi)}{\partial(y,u)} + u_y \frac{\partial(\phi,\psi)}{\partial(u,x)} = \frac{\partial(\phi,\psi)}{\partial(x,y)} \quad \cdots \quad (**)$$

Combining (*) & (**) we get

$$\underline{a u_x + b u_y = c}$$

This completes the proof.

Exm :- Solve : $x u_x + y u_y = u$. (Linear)

Characteristic eqns :

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}.$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} \Rightarrow \ln\left(\frac{y}{x}\right) = \ln(c_1)$$

$$\Rightarrow \phi(x,y,u) = \frac{y}{x} = c_1$$

$$\& \quad \frac{dy}{y} = \frac{du}{u} \Rightarrow \psi(x,y,u) = \frac{u}{y} = c_2$$

So the general solution is : $\underline{F\left(\frac{y}{x}, \frac{u}{y}\right) = 0}$

where F is an arbitrary fn.

H.W

Exm :- Solve : $x u_x + y u_y = nu$

Characteristic eqn :

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{nu}$$

$\phi(x,y,u) = \frac{y}{x} = c_1$, So the general soln

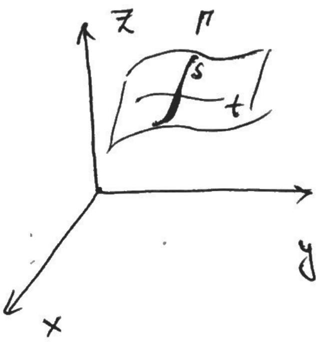
$\& \ \psi(x,y,u) = \frac{u}{y^n} = c_2$ is : $F\left(\frac{y}{x}, \frac{u}{y^n}\right) = 0$

H.W

Solve: $x^2 u_x + y^2 u_y = (x+y)u$

Ans: $F\left(x^{-1} - y^{-1}, \frac{x-y}{u}\right) = 0$ or $F\left(\frac{xy}{u}, \frac{x-y}{u}\right) = 0$

Problems on Cauchy data / IVP :-



A. ~~Unique Soln~~

1. $x u_x - y u_y = u$

$\Gamma: u = x^2$ on $y = x$, $1 \le y \le 2$

Parametrization of Cauchy data:

$$\Gamma: x(s) = s, \quad y(s) = s, \quad u(s) = s^2, \quad s \in [1,2] \quad \cdots (1)$$

Characteristic eqn:

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = -y, \quad \frac{du}{dt} = u$$

$$\Rightarrow x = c_1 e^t, \quad y = c_2 e^{-t}, \quad u = c_3 e^t \quad \cdots (11)$$

To get the whole integral surface, we take

$$x = X(s,t), \quad y = Y(s,t), \quad u = U(s,t)$$

W.L.O.G, let, at $t = 0$: the parametrization gives the curve Γ.

i.e. $X(s,0) = s$, $Y(s,0) = s$, $U(s,0) = s^2$

From (11) $\Rightarrow$ $c_1 = s$, $c_2 = s$, $c_3 = s^2$

Therefor, $X(s,t) = se^t$, $Y(s,t) = se^{-t}$, $U(s,t) = s^2 e^t$

$XY = s^2$, $\frac{X}{Y} = e^{2t}$

So, eliminating s & t, $U = XY\sqrt{\frac{X}{Y}}$

Hence, the soln is $u = xy\sqrt{\frac{x}{y}} = x\sqrt{xy}$.

As s varies we get diff. pt on the cauchy data, and as t varies we get diff ~~char~~ pt on a char. curve.

$$\begin{cases} x = X(s,t), \quad y = Y(s,t) \implies \begin{array}{l} s = S(x,y) \\ t = T(x,y) \end{array} \\ u = U(s,t) = u(x,y) \end{cases}$$

Need Inverse fn Theorem.

(✱)

<u>Inverse function Theorem :—</u>

Let, $G : U(\subseteq \mathbb{R}^n) \to \mathbb{R}^n$ be a C^1 function.

$G \equiv (G_1, G_2, G_3, \dots, G_n)$. Let,

$$J_G = \begin{pmatrix} \frac{\partial G_1}{\partial x_1} & \frac{\partial G_1}{\partial x_2} & \cdots & \frac{\partial G_1}{\partial x_n} \\ \frac{\partial G_2}{\partial x_1} & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \frac{\partial G_n}{\partial x_1} & \cdots & & \frac{\partial G_n}{\partial x_n} \end{pmatrix}$$

and

$|J_G| \neq 0$ at $\bar{x}_0$. Then, $\exists$ an open set $V \subset U$ with $\bar{x}_0 \in V$ and an open set $W \subseteq \mathbb{R}^n$ with $G(\bar{x}_0) \in W$ such that $G : V \to W$ is 1-1, onto and $G^{-1} : W \to V$ is C^1.

(✱): $G = (X(s,t), Y(s,t))$ with $\bar{x}_0 = (s, 0)$.

$$\begin{vmatrix} X_s & X_t \\ Y_s & Y_t \end{vmatrix}_{(s,0)} \neq 0 \quad \text{(Transversality Condition)}$$