H.W Nature of the eqn.

$$\operatorname{div}\left(\frac{\nabla u}{(1+|\nabla u|^2)^{1/2}}\right) = 0$$

$$\Rightarrow \quad \nabla \cdot \left(\frac{(u_x, u_y)}{(1+u_x^2+u_y^2)^{1/2}}\right) = 0$$

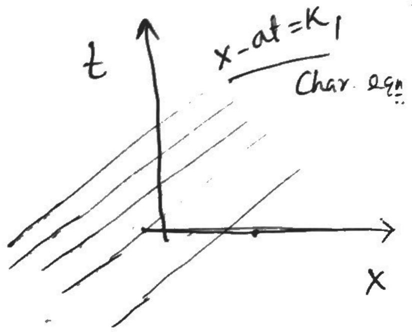
$$\Rightarrow \quad \frac{\partial}{\partial x}\left(\frac{u_x}{(1+u_x^2+u_y^2)^{1/2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{(1+u_x^2+u_y^2)^{1/2}}\right) = 0$$

$$\Rightarrow \quad \frac{(1+u_y^2)\,u_{xx} + (1+u_x^2)\,u_{yy}}{(1+u_x^2+u_y^2)^{3/2}} = 0$$

Q.L. of order 2

Exm: Transport eqn:-

$$u_t + a u_x = 0$$



Characteristic eqn:

$$\frac{dt}{1} = \frac{dx}{a} = \frac{du}{0}$$

$$\Rightarrow \quad x - at = c_1 \quad \& \quad u = c_2$$

So, the general eqn. is $u = f(x-at)$, for an arbitrary fn. f.

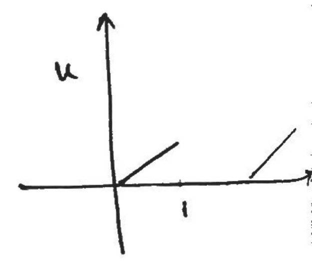
Now, if the initial condn. $u(x,0) = \phi(x)$

then, $u(x,0) = f(x) = \phi(x)$.

So, the particular integral would be:

$$u(x,t) = \phi(x-at)$$

Try $\phi(x) = \begin{cases} x, & 0<x<1 \\ 0, & \text{else} \end{cases}$



Exm: $2u_x + 3u_y + 8u = 0$ with Γ: $3x - 2y = 1$, $u(x,y) = e^{-4x}$

Characteristic eqn:

$$\frac{dx}{2} = \frac{dy}{3} = \frac{du}{-8u}$$

$$\Rightarrow \quad \phi(x,y,u) = 3x - 2y = c_1$$

$$\& \quad \psi(x,y,u) = u e^{4x} = c_2$$

So, the general soln is:

$$\psi = f(\phi)$$

i.e. $u(x,y) = e^{-4x} f(3x-2y)$

On Γ: $e^{-4x} = e^{-4x} f(1)$

$\Rightarrow$ $f(1) = 1$. These are infinitely many f with $f(1) = 1$

Hence, infinitely many soln.

H.W: Solve: $u_x = cu + d(x,y)$ $\quad \cdots (*)$ $\quad u(x,y) = e^{cx}\left(\int_0^x e^{-ct} d(t,y)\,dt + f(y)\right)$

i) Γ: $u(0,y) = y$. Show that, the Cauchy problem has a unique soln.

ii) Γ: $u(x,0) = e^{cx}$ & $d(x,y) = 0$. Show that $(*)$ has many soln

iii) Γ: $u(x,0) = \sin x$, $d(x,y) = 0$. Show that $(*)$ has no soln.

Exm

i) $u_x + x u_y = x^2 y$; $\Gamma: u(x, 2x + \frac{x^2}{2}) = 5x$

ii) $(y+u) u_x + y u_y = x - y$; $\Gamma: y > 0, -\infty < x < \infty$

& $u = 1 + x$ on $y = 1$.

i) $$\frac{dx}{1} = \frac{dy}{x} = \frac{du}{x^2 y}$$

$$\frac{x^2}{2} - y = K_1 \quad \& \quad \frac{dx}{1} = \frac{du}{x^2 y}$$

...

ii) Char. eqns:

$$\frac{dx}{y+u} = \frac{dy}{y} = \frac{du}{x-y}$$

$$\frac{d(x+u)}{x+u} = \frac{dy}{y} \Rightarrow (x+u)/y = c_1$$

& $$\frac{d(x-y)}{u} = \frac{du}{x-y} \Rightarrow (x-y)^2 = u^2 + c_2$$

~~Hence~~ $\cancel{(x-y)^2 - u^2 = f(x+u-y)}$ ~~is the general soln~~

~~But,~~ $\cancel{\phi(x,y,u) = x+u-y = c_1}$ & $\cancel{\psi(x,y,u) = (x-y)^2 - u^2 = c_2}$

~~are not l.i.~~ $\cancel{\begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix} = 0.}$

$$\cancel{\frac{d(u+y)}{x} = \frac{dx}{y+u} \Rightarrow (u+y)^2 - x^2 = c_3.}$$

~~The gen. soln is:~~

$$\cancel{(u+y)^2 - x^2 = f(x+u-y)}$$

The gen. solⁿ is:

$$(x-y)^2 - u^2 = f\left(\frac{x+u}{y}\right), \quad f \text{ is arbitrary.}$$

Γ: $y = 1$; $u = 1 + x$

$$\therefore \quad f(1+2x) = -4x$$

$$\therefore \quad f(x) = 2(1-x)$$

So, $$\underline{(x-y)^2 - u^2 = \frac{2}{y}(y-x-u)} \;.$$

* <u>Burger's Eqⁿ</u> :- (Inviscid)

$$u_t + u u_x = 0, \quad u(x,0) = \phi(x).$$

$$\left\{ \begin{array}{l} \text{Char. eqⁿ} \rightarrow \tau \text{ (instead of } t) \\ \Gamma : \; x = s, \; t = 0, \quad u = \phi(s) \end{array} \right.$$

Char. eqⁿ :

$$\left. \begin{array}{l} \dfrac{dt}{d\tau} = 1, \quad \dfrac{dx}{d\tau} = u, \quad \dfrac{du}{d\tau} = 0. \\ \text{I.C. : At } \tau = 0 \quad t(0) = 0, \quad x(0) = s, \quad u(0) = \phi(s). \end{array} \right\} \cdot (1)$$

The solution to (1) is ~~gen~~ given by

$$t = T(s,\tau) = \tau + k_1, \; u = U(s,\tau) = k_2$$

$$x = X(s,\tau) = k_2 \tau + k_3$$

This gives: $t = \tau$, $u = \phi(s)$ and $x = \phi(s)\tau + s$

$$J = \frac{\partial(x,T)}{\partial(s,\tau)} = \begin{vmatrix} x_s & x_\tau \\ T_s & T_\tau \end{vmatrix}$$

$$= \begin{vmatrix} \phi'(s)\tau + 1 & \phi(s) \\ 0 & 1 \end{vmatrix} = 1 + \tau\phi'(s)$$

Something is wrong at $1 + \tau\phi'(s) = 0$ i.e. $\underline{\tau = -\dfrac{1}{\phi'(s)}}$

~~Nb~~ However; $J\big|_{\tau=0} = 1 \neq 0$

So, we get unique soln near $(s, 0, \phi(s)) \in \Gamma$

i.e. $x = \phi(s)t + s$

$$\Rightarrow x = ut + s \Rightarrow s = x - ut$$

And, $u = \phi(s)$
$= \phi(x - ut)$

Hence, $u(x,t) = \phi(x - ut)$ given an implicit soln.

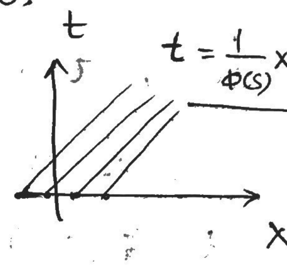
Characteristic ~~projection~~ lines: (curves)

$$\gamma_s : x = \phi(s)t + s$$

along which u has the const. value : $u = \phi(s)$.

~~Let~~, For a fixed s, γ_s represents the path of the particle located at $x = s$. ~~at time t=0~~.

$$t = \frac{1}{\phi(s)}x - \frac{s}{\phi(s)}$$



Now, γ_{s_1} & γ_{s_2} intersects at a point (x, t) with

$$t = -\frac{s_2 - s_1}{\phi(s_2) - \phi(s_1)}$$

A. ϕ is increasing means : $t < 0$ # so, no intersection.

B. ϕ is decreasing or non-increasing : u becomes singular for some $t > 0$.

$$u_x = \frac{\phi'(s)}{1 + \phi'(s)t}$$

So, for $\phi'(s) < 0$, we see that u_x becomes infinite at $t = -\dfrac{1}{k\phi'(s)}$

$T_0 = -\dfrac{1}{\phi'(s)}$ gives the blow up in the solution.

i.e. there can not exist a solution of class C^1 beyond time T_0. (Behavior of non-linear PDEs).

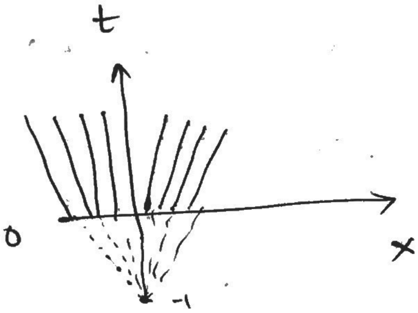
$\Rightarrow$ In literature, sometimes it is thus considered 'weak solution' of non-linear eqn, dropping the smoothness property of the solution!

<u>Exm. 1.</u> Let, $\phi(x) = x$ i.e. $\phi \uparrow$

The char. eqn are given by:

$$x = st + s$$

$$\Rightarrow t = \frac{1}{s}x - 1$$



The solution is: $u = \dfrac{x}{t+1} \quad \forall t > 0$

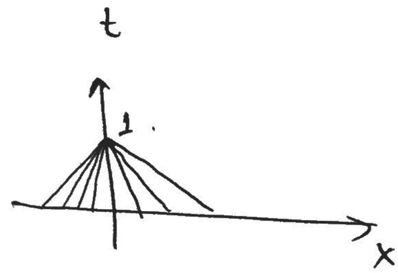
and it never breaks.

The char. ~~eqn~~ lines never intersect.

<u>Exm 2</u> Let, $\phi(x) = -x$ i.e. $\phi \downarrow$

Char. eqn: $x = -st + s$

$$\Rightarrow t = -\frac{1}{s}x + 1$$



The solution is: $u = \dfrac{x}{t-1}$

The solution blows up at $t = 1$, and the char. lines intersects at $t = 1$.