Non-linear PDEs :-

Fully non-linear PDE:

$$f(x, y, u, u_x, u_y) = 0$$

Notation: $p = u_x$, $q = u_y$.

Assumption :-

i) f is twice continuously differentiable in x, y, u, u_x, u_y;

ii) $f_p^2 + f_q^2 \neq 0$

The geometry for non-linear is more involved than in the quasi-lin. case.

Existence of the solution is guranteed by Cauchy-Kloweu theorem.

~~$P(x_0, y_0, z_0)$~~

Unlike quasi-linear case, we do not have a char. direction. Let, $P(x_0, y_0, z_0)$ be a point on the integral surface $z = u(x, y)$. The eqn of the tangent plane at P is:

$$u_x(x_0, y_0)(x - x_0) + u_y(x_0, y_0)(y - y_0) = z - z_0 \quad \cdots (\mathrm{I})$$

Since, u is unknown, $p_0 = u_x(x_0, y_0)$ & $q_0 = u_y(x_0, y_0)$ are also unknown. But they satisfy:

$$f(x_0, y_0, z_0, p_0, q_0) = 0. \quad \cdots (\mathrm{II})$$

(I) & (II) determine a one-parameter family of planes passing through $P(x_0, y_0, z_0)$ and one of them is the tangent plane to $z = u(x, y)$ at $P(x_0, y_0, z_0)$.

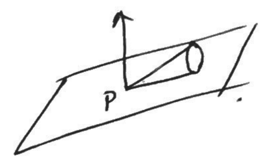
$$\begin{cases} H(x,y,z,p,q) = z - z_0 - p(x-x_0) - q(y-y_0) & \cdots (\text{III}) \\ \& \quad f(x_0,y_0,z_0,p,q) = 0 & \cdots (\text{IV}). \end{cases}$$

Consider the envelope of the family of planes described by (III) & (IV), such that:

~~a)~~ Envelope touches each member of the family of planes.

~~b)~~

All such planes pass ~~to~~ through $P(x_0, y_0, z_0)$ and thus envelope a cone with vertex P. These cones are called <u>Monge Cone</u>.



$\Rightarrow$ (IV) $\Rightarrow$ $q = G(x_0, y_0, z_0, p)$

(III) $\Rightarrow$ $H(x, y, z, p, G(p)) = 0$

i.e. $z - z_0 = p(x - x_0) + G(p)(y - y_0) \quad \cdots (\text{V})$

Now, take derivative of (IV) & (V) w.r.to p to get

$$0 = (x - x_0) + G'(p)(y - y_0) \quad \cdots (\text{VI})$$

$$0 = f_p + G'(p) f_q \quad \cdots (\text{VII})$$

Eliminating $G'(p)$ from (VI) & (VII), we get

$$f_q(x - x_0) - f_p(y - y_0) = 0 \quad \cdots (\text{VIII})$$

So, we have from (V) & (VIII)

$$\frac{x - x_0}{+f_p} = \frac{y - y_0}{+f_q} = \frac{z - z_0}{pf_p + qf_q} \quad \cdots (\text{IX})$$

(ix) Corresponds to the eqn of lines that generate the 'Monge Cone' as we vary p. The direction of such a line can be taken as a characteristic direction.

Let, $(x(t), y(t), u(t))$ be a curve through $P(x_0, y_0, z_0)$ on $z = u(x,y)$. Therefore, the characteristic system would be:

$$\left.\begin{aligned} \frac{dx}{dt} &= f_p, \quad \frac{dy}{dt} = f_q \\ \frac{du}{dt} &= pf_p + qf_q. \end{aligned}\right\} \text{with} \quad \cdots (*)$$

$$x(0) = x_0, \quad y(0) = y_0, \quad u(0) = z_0.$$

(*) is under-determined, as p & q are also unknown.

$$f(x, y, u, p, q) = 0$$

Take derivative w.r.to x & y.

$$\left.\begin{aligned} & f_x + f_u p + f_p \cdot p_x + f_q \cdot q_x = 0 \\ \& \quad & f_y + f_u q + f_p \cdot p_y + f_q \cdot q_y = 0 \end{aligned}\right\} \cdots (**)$$

Now along a char. curve,

$$\frac{dp}{dt} = p_x \frac{dx}{dt} + p_y \cdot \frac{dy}{dt} \qquad [p_y = q_x]$$

$$\left.\begin{aligned} &= -f_x - f_u p. \qquad \text{from } (**) \\ \text{Similarly, } \quad \frac{dq}{dt} &= -f_y - q f_u \end{aligned}\right\} (***)$$

(*) & (***) together form characteristic equations or Charpit's Method.

A solution of the char. eqns is known as characteristic Strip.

Theorem: The function $f(x, y, u, p, q)$ is constant along every char. strip of the eqn $f(x, y, u, p, q) = 0$.

Proof:- Along a char. strip, we have

$$\frac{d}{dt} f(x(t), y(t), u(t), p(t), q(t)) = f_x x' + f_y y' + f_u u' + f_p p' + f_q q'$$

$$= f_x f_p + f_y f_q + f_u (p f_p + q f_q) - f_p (f_x + f_u p) - f_q (f_y + q f_u)$$

$$= 0.$$

Therefore, $f(x, y, u, p, q)$ is constant along the strip.

Cauchy problem:- An integral surface $z = u(x, y)$ contains an initial curve Γ:
$x = g(s)$, $y = h(s)$, $u = J(s)$.

~~Find~~ Steps for solution:

To find $x = X(s,t)$, $y = Y(s,t)$, $u = U(s,t)$
$p = \Phi(s,t)$, $q = \Psi(s,t)$ such

that, ~~at t=0~~:

$$\frac{dx}{dt} = f_p, \quad \frac{dy}{dt} = f_q, \quad \frac{du}{dt} = p f_p + q f_q$$

$$\frac{dp}{dt} = -f_x - p f_u, \quad \frac{dq}{dt} = -f_y - q f_u$$

and at $t = 0$: $x = g(s)$, $y = h(s)$, $u = J(s)$, $p = \phi(s)$, $q = \psi(s)$

~~$f(g(s), h(s)$~~, $f(g(s), h(s), J(s), \phi(s), \psi(s)) = 0$

and $\frac{du}{ds} = J'(s) = p g'(s) + q h'(s)$.

At the end eliminate s & t to get $z = u(x,y)$, provided

$$\begin{vmatrix} X_s & X_t \\ Y_s & Y_t \end{vmatrix} = \begin{vmatrix} g'(s) & f_p \\ h'(s) & f_q \end{vmatrix} = g'(s) f_q - f_p h'(s) \neq 0$$

Exm: Solve: $pq = u$, $u(0,y) = y^2$

Let, $f(x,y,u,p,q) = pq - u$

The char. eqns are:

$$\frac{dx}{dt} = f_p = q, \quad \frac{dy}{dt} = f_q = p, \quad \frac{du}{dt} = pf_p + qf_q = 2pq$$

$$\frac{dp}{dt} = -f_x - pf_u = p, \quad \frac{dq}{dt} = -f_y - qf_u = q.$$

So,

$$\left.\begin{aligned} & p(t) = ce^t, \quad q(t) = de^t \\ & x(t) = de^t + d_1 \quad \& \; y(t) = ce^t + c_1 \\ & \& \quad u = cde^{2t} + K. \end{aligned}\right\} \cdots (*)$$

Now, on Γ: at $t = 0$.

$x = 0$, $y = s$, $u = s^2$

$p(s) q(s) = u(s) = s^2$

and $\frac{du}{ds} = 2s = p(s)\frac{dx}{ds} + q(s)\frac{dy}{ds}$

$= q(s)$

$\therefore \; p(s) = s/2.$

Let, $x = X(s,t)$, $y = Y(s,t)$, $u = U(s,t)$, $p = \phi(s,t)$

$q = \psi(s,t)$ is a char. strip.

$\cancel{0 = d + d_1}$ $c = s/2$, $d = 2s$, $d_1 = -d = -2s$

$c_1 = s/2$, $K = 0$.

Therefore we have:

$$X(s,t) = 2s(e^t - 1), \quad Y(s,t) = \frac{s}{2}(e^t + 1)$$

$$U(s,t) = s^2 e^{2t}$$

$$\phi(s,t) = \frac{s}{2} e^t, \quad \psi(s,t) = 2se^t.$$

Also,

$$\begin{vmatrix} X_s & X_t \\ Y_s & Y_t \end{vmatrix} = \begin{vmatrix} 2(e^t - 1) & 2se^t \\ \frac{1}{2}(e^t+1) & \frac{s}{2}e^t \end{vmatrix}$$

$$= s(\cancel{e^{2t}} - e^t) - s(\cancel{e^{2t}} + e^t)$$

$$= -2se^t \neq 0.$$

$$x + 4y = 4se^t, \quad u = s^2 e^{2t}$$

So, $u = \left[\frac{1}{4}(x+4y)\right]^2$ is the soln.

Solve: $2p^2 x + qy = u, \quad u(x,1) = \frac{x}{2}.$

[Ans: $u = x/2$]

Let, $f(x,y,u,p,q) = 2p^2 x + qy - u.$

The char. eqns:

$$\frac{dx}{dt} = f_p = 4px, \quad \frac{dy}{dt} = f_q = y, \quad \frac{du}{dt} = 4p^2 x + qy$$

$$\frac{dp}{dt} = -f_x - pf_u = -2p^2 + p, \quad \frac{dq}{dt} = -f_y - f_u \cdot q$$

$$= -q + q = 0$$

H.W. Solve: $pq = xy, \quad u(x,0) = x.$

Ans: $u^2 = x^2(1+y^2)$

$X(s,t) = s\cosh t$

$Y(s,t) = \sinh t$

$U(s,t) = s\cosh^2 t$

$p(s,t) = \cosh t$

$q(s,t) = s\sinh t.$

Cauchy - Kowalewski Theorem :-

$$a(x,y,u)\, u_x + b(x,y,u)\, u_y = c(x,y,u) \quad \cdots (1)$$

$$\Gamma: \; x = f(s), \; y = g(s), \; z = h(s), \quad s \in I$$

Version 1 :- ~~Let $\Gamma(s) = (x(t), y(t))$~~

Let, $\Gamma \left((f(s), g(s), h(s)) \text{ with } h(s) = u(f(s), g(s)) \right)$ be a non-characteristic curve i.e.

$$\begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix} = b f'(s) - a g'(s) \neq 0 . \text{ and}$$

a, b, c are C^1 functions. Then, the cauchy problem (1) has a solution in a neighbourhood of the initial curve Γ. The solution is locally unique near Γ.

Version 2 :- Consider $\frac{du}{dx} = F(x, y, u, u_y)$, with $\cdot$(II)

$$u(x_0, y) = g(y). \quad g(y_0) = z_0, \; g'(y_0) = q_0$$

Suppose g is a C^1 function in a neighbourhood of y_0. Also, F is continuous and its derivatives are cont. near (x_0, y_0, z_0, q_0). Then $\exists$ a unique fn $\phi(x,y)$ s.t.

I) ϕ is C^1 near (x_0, y_0)

II) ϕ satisfies (II) near (x_0, y_0)

III) In a nbd of y_0, $\phi(x_0, y) = g(y)$.