Second Order PDEs

It deals with the classification of 2nd order PDEs and their Canonical forms.

Quasi-linear PDEs:

$$a(x,y,u,u_x,u_y)\,u_{xx} + 2b(x,y,u,u_x,u_y)\,u_{xy} + c(x,y,u,u_x,u_y)\,u_{yy} + d(x,y,u,u_x,u_y) = 0.$$

Linear PDEs :-

$$a(x,y)\,u_{xx} + 2b(x,y)\,u_{xy} + c(x,y)\,u_{yy} + d(x,y)\,u_x + e(x,y)\,u_y + f(x,y)\,u + g(x,y) = 0 \quad \ldots (*)$$

The classification of PDEs is suggested by the classification of conic section:

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0 \ldots$$

It represents hyperbola, parabola or ellipse as $b^2 - ac$ is +ve, 0 or -ve.

Prototype or standard eqn :-

Parabolic : $u_t - u_{xx} = 0$ (Heat)

Hyperbolic : $u_{tt} - u_{xx} = 0$ (Wave)

Elliptic : $u_{xx} + u_{yy} = 0$ (Laplace)

A ~~Many~~ homogeneous linear 2nd-order PDEs can be transformed to one of the above by making a change of variables.

Theorem 5:-

With a change of ~~variables~~ co-ordinates from (x,y) to (ξ,η), the eqn. (*) transforms to:

$$A(\xi,\eta)\,\omega_{\xi\xi} + 2B(\xi,\eta)\,\omega_{\xi\eta} + C(\xi,\eta)\,\omega_{\eta\eta} = \Psi(\omega_\xi, \omega_\eta, \omega, \xi, \eta) \quad \ldots (**)$$

where,

$$A(\xi,\eta) = a\xi_x^2 + 2b\,\xi_x\xi_y + c\,\xi_y^2$$

$$B(\xi,\eta) = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\,\xi_y\eta_y$$

$$C(\xi,\eta) = a\eta_x^2 + 2b\,\eta_x\eta_y + c\,\eta_y^2.$$

Moreover
$$B^2 - AC = \left[\frac{\partial(\xi,\eta)}{\partial(x,y)}\right]^2 (b^2 - ac),$$ so that the form of the PDE remains invariant.

Proof:-

Let,
$$\omega(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta)),$$ such that
$$u(x,y) = \omega(\xi(x,y), \eta(x,y)).$$

Now,
$$u_x = \omega_\xi\,\xi_x + \omega_\eta\,\eta_x$$
$$u_y = \omega_\xi\cdot\xi_y + \omega_\eta\cdot\eta_y.$$

Differentiating once more,

$$u_{xx} = \omega_{\xi\xi}\,\xi_x^2 + 2\omega_{\xi\eta}\,\xi_x\eta_x + \omega_{\eta\eta}\,\eta_x^2 + \text{l.o.t}$$

$$u_{xy} = \omega_{\xi\xi}\,\xi_x\xi_y + \omega_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + \omega_{\eta\eta}\,\eta_x\eta_y + \omega_\xi\,\xi_{xy} + \omega_\eta\,\eta_{xy}$$

$$u_{yy} = \omega_{\xi\xi}\,\xi_y^2 + 2\omega_{\xi\eta}\,\xi_y\eta_y + \omega_{\eta\eta}\,\eta_y^2 + \text{l.o.t}.$$

Therefore, the eqn (*) becomes:

$$A\omega_{\xi\xi} + 2B\omega_{\xi\eta} + C\omega_{\eta\eta} = \psi(\omega_\xi, \omega_\eta, \omega, \xi, \eta)$$

where, $A(\xi,\eta)$, $B(\xi,\eta)$, $C(\xi,\eta)$ are as the required forms. Also,

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix}.$$

Now, taking the determinant on both sides

$$B^2 - AC = (\xi_x \eta_y - \xi_y \eta_x)^2 (b^2 - ac).$$

Hence the proof.

Defn:- The PDE (*) is

i) hyperbolic if $b^2 - ac > 0$

ii) parabolic if $b^2 - ac = 0$

iii) Elliptic if $b^2 - ac < 0$

Canonical Forms :-

Since A and C have the same form, we may choose ξ & η such that

$$A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0 \quad \text{and}$$

$$C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0$$

Now, $a\left(\frac{\xi_x}{\xi_y}\right)^2 + 2b\left(\frac{\xi_x}{\xi_y}\right) + c = 0$ is the common eqn for ξ and η.

Now, along the curve $\xi(x,y) = K$, we have

$$d\xi = \xi_x dx + \xi_y dy = 0 \quad \text{i.e.} \quad \frac{dy}{dx} = -\frac{\xi_x}{\xi_y}.$$

So, $a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c = 0$

$$\therefore \frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \lambda^{\pm}$$

~~Since $b^2-ac>0$~~

Hyperbolic PDEs

Let, $b^2 - ac > 0$, then, we get two distinct families of characteristic curves:

$$\xi(x,y) = y - \lambda^{+}x = c_1 , \quad \eta(x,y) = y - \lambda^{-}x = c_2$$

The reduced eqn would be: $(B \neq 0)$

$$\omega_{\xi\eta} + F(\omega_{\xi}, \omega_{\eta}, \omega, \xi, \eta) = 0 \quad (1)$$

This is the first canonical form of hyperbolic eqn.

\# If $\alpha = \xi + \eta$, $\beta = \xi - \eta$, then, (1) becomes:

$\omega_{\alpha\alpha} - \omega_{\beta\beta} + \dots = 0$, which is the second canonical form of the hyperbolic eqn.

EXM: $u_{tt} - k^2 u_{xx} = 0$.

$a = 1$, $b = 0$, $c = -k^2$

$b^2 - ac = k^2 > 0$. $\therefore$ Hyperbolic

$a\alpha^2 - 2b\alpha + c = 0$

$\alpha^2 - k^2 = 0$

$\alpha = \pm k$

$\frac{dx}{dt}$ ~~$\frac{dt}{dx}$~~ $= \pm k \Rightarrow$ ~~$\xi = y - kx$~~ $\xi = x - kt$

~~$\eta = y + kx$~~. $\eta = x + kt$.

The eqn becomes:

$$u_{\xi\eta} = 0 \Rightarrow u_{\xi} = f(\eta)$$

$$\therefore u = f(\eta) + g(\xi).$$

$\therefore$ ~~u(x,y~~

$u(x,t) = f(x-ct) + g(x+ct)$, f & g are arbitrary fn.

Parabolic PDEs

Let, $b^2 - ac = 0$. Then, one charac. family of eqn. is:

$$\frac{dy}{dx} = \frac{b}{a}$$

$$\Rightarrow \xi(x,y) = ay - bx = C_1$$

We have to choose $\eta(x,y)$ arbitrarily s.t. ($\eta = x$ or y works)

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0.$$

The reduced eqn. is:

$$u_{\eta\eta} + \Phi(\xi, \eta, u, u_\xi, u_\eta) = 0$$

This is called the canonical form of the parabolic PDE.

Exm :- $x^2 u_{xx} - 2xy\, u_{xy} + y^2 u_{yy} + x u_x + y u_y = 0$.

$a = x^2$, $b = -xy$, $c = y^2$; $b^2 - ac = 0$. $\therefore$ parabolic.

$\therefore$

$$\frac{dy}{dx} = \frac{b}{a} = -\frac{xy}{x^2} = -\frac{y}{x}.$$

$$\Rightarrow \xi(x,y) = xy = C_1$$

We take, $\eta(x,y) = x$. $\quad \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} y & x \\ 1 & 0 \end{vmatrix} = -x \neq 0$

The canonical form is:

$$\eta^2 u_{\eta\eta} + \eta u_\eta = 0$$

Elliptic PDEs

Let, $b^2 - ac < 0$. Then

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a}$$ has no real

solutions, instead has two complex conjugate solutions. ξ and η.

Let, $\alpha = \frac{\xi + \eta}{2}$, $\beta = \frac{\xi - \eta}{2i}$ so that

$\xi = \alpha + i\beta$, $\eta = \alpha - i\beta$.

Then, the eqn reduces to:

$$u_{\alpha\alpha} + u_{\beta\beta} + \Phi(u_\alpha, u_\beta, u, \alpha, \beta) = 0,$$

which is the canonical form of the elliptic PDEs.

<u>Exm</u>: $u_{xx} + x^2 u_{yy} = 0$

$a = 1,\ b = 0,\ c = x^2$

$b^2 - ac = -x^2 < 0$: Elliptic.

We have, $\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \pm ix$

$\Rightarrow\ \xi = y - i\frac{x^2}{2} = C_1 \quad \eta = y + i\frac{x^2}{2} = C_2$.

Take: $\alpha = y$, $\beta = -\frac{x^2}{2}$.

$u_x = -u_\beta . x$

$u_y = u_\alpha$

$u_{xx} = -u_\beta - x[u_{\alpha\beta}.0 \mp u_{\beta\beta}x$

$u_{yy} = u_{\alpha\alpha}$

The canonical form is:

$$-u_\beta + x^2 u_{\beta\beta} + x^2 u_{\alpha\alpha} = 0$$

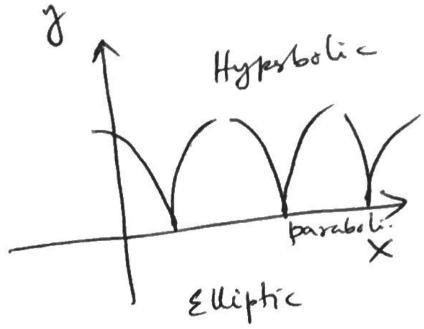
$$\Rightarrow\ u_{\alpha\alpha} + u_{\beta\beta} = \frac{u_\beta}{x^2} = -\frac{u_\beta}{2\beta}.$$

Exm (Tricomi Eq$\underline{n}$)

$$u_{yy} - y\,u_{xx} = 0$$

$$a = 1, \quad b = 0, \quad c = -y$$

$$b^2 - ac = y$$



On the x-axis ($y=0$) : parabolic

for $y>0$, : Hyperbolic

for $y<0$: Elliptic.

$$\frac{dx}{dy} = \frac{b \pm \sqrt{b^2-ac}}{a} = \pm\sqrt{y}$$

$$\therefore \quad \xi = 3x - 2y^{3/2}, \quad \eta = 3x + 2y^{3/2}$$

The eq$\underline{n}$ reduces for $y>0$.

$$u_{\xi\eta} - \frac{1}{6}\,\frac{u_\xi - u_\eta}{\xi - \eta} = 0$$

H.W $\quad u_{xx} + u_{xy} - 2u_{yy} + 1 = 0 \quad$ in $0 \le x \le 1$, $y>0$

with $u = u_y = x$ on $y = 0$

H.W $\quad u_{xx} - 4u_{xy} + 4u_{yy} = e^y$

$$a=1, \quad b=-2, \quad c=4, \quad b^2-ac = 4-4 = 0 : \text{parabolic}$$

$$\frac{dx}{dy} = + \frac{b}{a} = -2 \quad \Rightarrow \quad x + 2y = c_1 = \xi$$

$$\eta = x.$$

$$\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = -2 \neq 0.$$

The eq$\underline{n}$ reduces to:

$$\cancel{u_\xi}\; u_{\eta\eta} = e^{\xi - \eta/2}$$

H.W $u_{xx} + (1+y^2)u_{yy} - 2y(1+y^2)u_y = 0$

$(w_{\xi\xi} + w_{\eta\eta} - 4\tanh\eta \, . \, w_\eta = 0)$

$\xi = x, \ \eta = \tan^{-1} y$

General Classification :-

A second order linear PDE in n variables is of the form:

$$\sum_{i,j}^{n} a_{ij}(\bar{x}) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\bar{x}) \frac{\partial u}{\partial x_i} + c(\bar{x})\, u + d(\bar{x}) = 0$$

$\bar{x} = (x_1, x_2, \dots, x_n)$. Let, $A = (a_{ij})_{n \times n}$. We may take A to be symmetric as we can adjust

$$a_{ij} = a_{ji}.$$

It is not usually possible to reduce the eqn to a canonical form. For convenience, we take only eqn with constant co-eff. Then, the reduction is possible.

i.e. $a_{ij}(\bar{x}) = a_{ij}$

and $b_i(\bar{x}) = b_i$, $c(\bar{x}) = c$, $d(\bar{x}) = d$.

$\Rightarrow$ Since A is symmetric, it is diagonalizable, i.e. $\exists$ an orthogonal matrix Q s.t.

$$Q^T A Q = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$$

If q_i is the ith column of Q, then the change of variable would be:

$$\eta_i(x_1, x_2, \dots, x_n) = q_i^T \cdot \bar{x}$$

i.e. $\bar{\eta} = Q^T \bar{x}$.

W.R.To the new co-ordinates, the principal part of the PDE takes the form:

$$\sum_{i=1}^{n} \lambda_i \frac{\partial^2 w}{\partial \eta_i^2}$$

Thm 6. The PDE is

a) elliptic if all the eigen-values of A are of same sign.

Exm: $u_{xx} + u_{yy} + u_{zz} = 0$

b) parabolic if any one of eigenvalues λ is zero, and others are of same sign.

Exm: $u_t - (u_{xx} + u_{yy} + u_{zz}) = 0$.

c) hyperbolic if all λ_k are non-zero and have the same sign except for precisely one of them.

Exm: $u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0$

Very Rare in practice

d) ultra-hyperbolic if all λ_k are non-zero and there are at least two of each sign.

Exm: $$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = \frac{\partial^2 u}{\partial x_3^2} + \frac{\partial^2 u}{\partial x_4^2}.$$

H.W. $y u_{xx} - 2u_{xy} + x u_{yy} = 0$. (parabolic: $xy = 1$
elliptic: $xy > 1$
hyperbolic: $xy < 1$

Exm: Classify: $u_{xx} - u_{xy} + u_{yy} = f(x,y)$

$$A = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}$$

$\lambda = \frac{1}{2}, \frac{3}{2}$: Elliptic.