# Hyperbolic PDEs

**Application:** Vibrating string (1D) or Vibrating membrane (2D)

(Strauss for deduction)

$$(1)\ldots \begin{cases} u_{tt} - c^2 u_{xx} = \phi(x,t) & \text{[Wave equation]} \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

Since the operator $\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right)$ is linear, we may write

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 \\ v(x,0) = f(x) \\ v_t(x,0) = g(x) \end{cases} \quad \ldots (\text{II})$$

$$\begin{cases} \omega_{tt} - c^2 \omega_{xx} = \phi(x,t) \\ \omega(x,0) = 0 \\ \omega_t(x,0) = 0 \end{cases} \quad \ldots (\text{III})$$

with $u(x,t) = v(x,t) + \omega(x,t)$.

We first solve:
$$\left.\begin{aligned} v_{tt} - c^2 v_{xx} &= 0 \\ v(x,0) &= f(x) \\ v_t(x,0) &= g(x) \end{aligned}\right\} \ldots (\text{II}).$$

$a = 1,\ b = 0,\ c = -c^2 \Rightarrow b^2 - ac = c^2 > 0$ : hyperbolic

The char. eqn be:

$$\frac{dx}{dt} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \pm c.$$

Char. curves:

$$\xi(x,t) = x - ct, \qquad \eta(x,t) = x + ct.$$

By chain rule,

$$v_x = v_\xi \xi_x + v_\eta \eta_x = v_\xi + v_\eta$$

$$v_t = -c v_\xi + c v_\eta.$$

Also,

$$v_{tt} - c^2 v_{xx} = (\partial_t - c\partial_x)(\partial_t + c\partial_x) v$$

$$= (-2c\partial_\xi)(2c\partial_\eta) v = 0$$

As, $c \neq 0$, the eqn. (II) reduces to:

$$v_{\xi\eta} = 0.$$

$$\Rightarrow \quad v_\xi = \tilde{F}(\xi).$$

$$\therefore \quad v = \int_{\xi_0}^{\xi} \tilde{F}(\xi)\, d\xi + G(\eta)$$

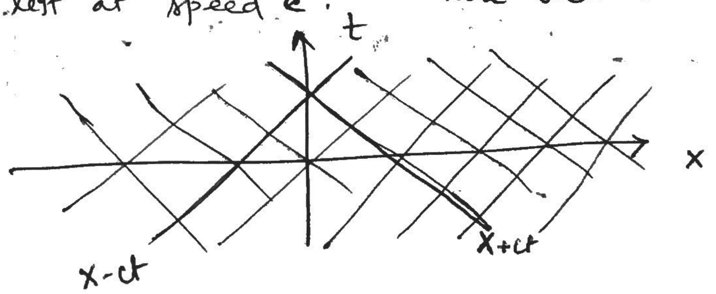
$$\therefore \quad v(\xi, \eta) = F(\xi) + G(\eta),$$ F and G are arbitrary fn. of single variable.

Therefore, the general soln. is:

$$v(x,t) = F(x-ct) + G(x+ct).$$

Now, $u \in C^2$ iff $F, G \in C^2$.

$F(x-ct)$ is a shape traveling to right at speed $c$, $G(x+ct)$ is traveling to left at speed $c$. These are traveling waves.



Any solution of the wave eq$\underline{\text{n}}$ is a superposition of forward and backward moving waves.

Now, IC: $u(x,0) = f(x)$, $u_t(x,0) = g(x)$.

So, $F(x) + G(x) = f(x) \quad \cdots (*)$

and $-F'(x) + G'(x) = \frac{1}{c} g(x) \cdots (**)$

Diff. $(*)$ w.r.to $x$,

$$F'(x) + G'(x) = f'(x)$$
$$-F'(x) + G'(x) = \frac{1}{c} g(x)$$

So, $G'(x) = \frac{1}{2} f'(x) + \frac{1}{2c} g(x)$

$$\therefore G(x) = \frac{f(x)}{2} + \frac{1}{2c}\int_0^x g(s)\,ds + A$$

& $F'(x) = \frac{1}{2} f'(x) - \frac{1}{2c} g(x)$

$$\therefore F(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_0^x g(s)\,ds + B$$

$(*) \Rightarrow \quad F(x) + G(x) = f(x)$

So, $A + B = 0$.

Therefore, the sol$\underline{\text{n}}$ is:

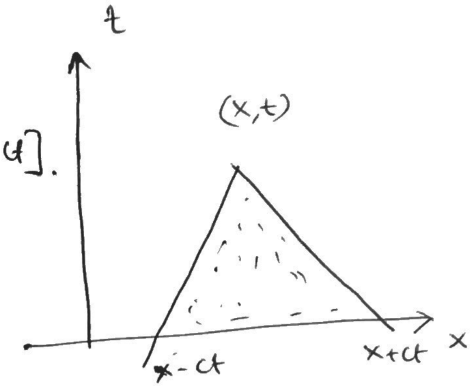
$$u(x,t) = F(x-ct) + G(x+ct)$$
$$= \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\left[\int_0^{x+ct} g(s)\,ds + \int_{x-ct}^{0} g(s)\,ds\right]$$

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds \quad \cdots (\#)$$

If, $f \in C^2$, $g \in C^1$, $u \in C^2$.

The solution formula is due to d'Alembert.

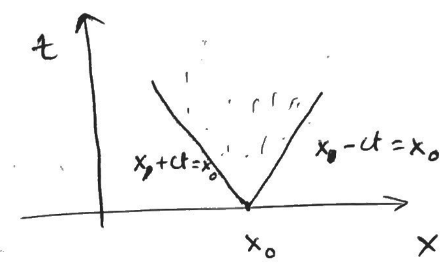
\# $v(x,t)$ is determined uniquely by the values of $f$, $g$ in $[x-ct, x+ct]$. This interval represents the <u>domain of dependence</u>.



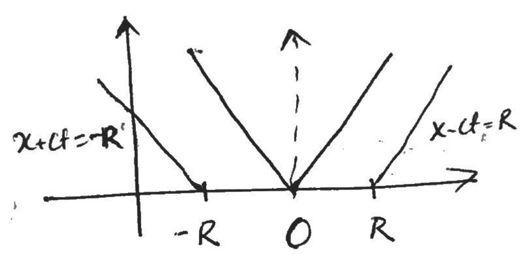

An initial condition ($f$ or $g$) at a point $(x_0, 0)$ can affect the soln for $t > 0$ in the region

$$x_0 - ct < x < x_0 + ct$$

This is called the domain of influence of the pt. $(x_0, 0)$.




$\Rightarrow$ If $f$ and $g$ vanish for $|x| > R$ then $u(x,t) = 0$ for $|x| > R + ct$




<u>Theorem 7</u> Let, $f \in C^2(\mathbb{R})$, $g \in C^1(\mathbb{R})$. Then, the soln $v(x,t)$ in (*) is unique.

[Pdes-course]

$\Rightarrow$ We want to solve (III): non-homogeneous wave eq$\underline{n}$

## The Duhamel principle

$$\begin{cases} w_{tt} - c^2 w_{xx} = \phi(x,t) \\ w(x,0) = 0 \\ w_t(x,0) = 0 \end{cases} \quad \cdots \text{(III)}$$

Analogus to Variation of parameter for ODEs. (See site)

(1) Let us consider the ~~four~~ following Cauchy problems with parameter $\tau > 0$. let, $v(x,t,\tau)$ satisfies the eq$\underline{n}$:

$$(*) \cdots \begin{cases} v_{tt} - c^2 v_{xx} = 0, \quad x \in \mathbb{R}, \; t > \tau \\ v(x,\tau;\tau) = 0, \\ v_t(x,\tau;\tau) = \phi(x,\tau) \end{cases}$$

Then, by d'Alembert, the sol$\underline{n}$ of $(*)$ is: $(f \in C^1)$

$$v(x,t;\tau) = \frac{1}{2c} \int_{x-c(t-\tau)}^{x+c(t-\tau)} \phi(s,\tau)\, ds. \quad \left[ \xi = t-\tau \right]$$

**Lemma 8** Let $f$ is a smooth f$\underline{n}$ of $x,t,\tau$ and

$$F(x,t) = \int_0^t \phi(x,t;\tau)\, d\tau. \text{ Then,}$$

$$\frac{\partial F}{\partial t} = f(x,t;t) + \int_0^t \phi_t(x,t;\tau)\, d\tau.$$

**Proof:**
$$\frac{\partial F}{\partial t} = \lim_{h \to 0} \frac{F(x,t+h) - F(x,t)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \left[ \int_0^{t+h} \phi(x,t+h;\tau)\, d\tau - \int_0^t \phi(x,t;\tau)\, d\tau \right]$$

$$= \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \phi(x, t+h; \tau)\, d\tau + \lim_{h\to 0} \int_0^t \frac{\phi(x, t+h; \tau) - \phi(x, t; \tau)}{h}\, d\tau$$

$$= I_1 + I_2 .$$

By Mean Value thm of integral calculus,

$$\int_t^{t+h} \phi(x, t+h; \tau)\, d\tau = h.\phi(x; t+h; t+\theta h), \quad \theta \in (0,1).$$

So, $I_1 = \lim_{h\to 0} \phi(x, t+h; t+\theta h) = \phi(x, t; t)$, $f \in C^1$.

And $I_2 = \int_0^t \phi_t (x, t; \tau)\, d\tau$ as $\phi \in C^1$.

Hence the result

② **Theorem 9** (Duhamel Principle)

$$w(x,t) = \int_0^t v(x, t; \tau)\, d\tau, \quad t \geq 0$$ is the

soln of (III).

Proof :- $\phi \in C^1$, so by d'Alembert $w \in C^2$.

Now, $w_t(x,t) = v(x,t;t) + \int_0^t v_t(x,t;\tau)\, d\tau$.

$$= \int_0^t v_t(x,t;\tau)\, d\tau .$$

$$w_{tt}(x,t) = v_t(x,t;t) + \int_0^t v_{tt}(x,t;\tau)\, d\tau .$$

$$= \phi(x;t) + \int_0^t v_{tt}(x,t;\tau)\, d\tau .$$

Now, $c^2 w_{xx}(x,t) = \int_0^t c^2 v_{xx}(x,t;\tau)\, d\tau. = \int_0^t v_{tt}(x,t;\tau)\, d\tau$

Thus, $\omega_{tt} - c^2\omega_{xx} = \phi(x,t)$.

Also, $\omega(x,0) = 0 = \omega_t(x,0)$

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So, the complete soln. of

$$\begin{cases} u_{tt} - c^2 u_{xx} = \phi(x,t) , & x \in \mathbb{R} \\ u(x,0) = f(x) & \cdots \quad (1) \\ u_t(x,0) = g(x) \end{cases}$$

is :

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

$$+ \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} \phi(s,\tau)\,ds\,d\tau.$$

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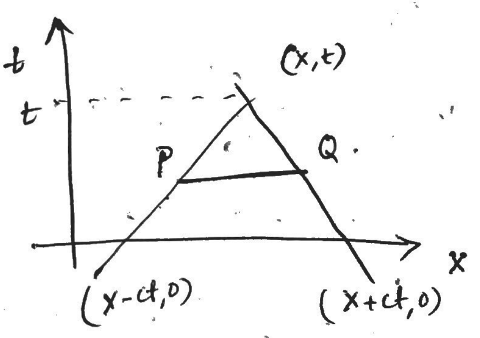
Integrating $\phi$ over the past history of the pt. $(x,t)$ back to the initial time $t=0$.



## Stability

Given $\epsilon > 0$, $\exists\, \delta > 0$ s.t. for every pair of data triples $(\phi_1, f_1, g_1)$ and $(\phi_2, f_2, g_2)$ satisfying
$\left( \phi \in C(\mathbb{R}\times[0,T]),\ \phi_x \in C(\mathbb{R}\times(0,T)),\ f \in C^2(\mathbb{R}),\ g \in C^1(\mathbb{R}) \right)$

$$|f_1(x) - f_2(x)| < \delta, \quad |g_1(x) - g_2(x)| < \delta \quad \forall x \in \mathbb{R}$$

and $|\phi_1(x,t) - \phi_2(x,t)| < \delta \quad \forall (x,t) \in \mathbb{R}\times[0,T]$,

the solns $u_1$ & $u_2$ of (1) satisfy:

$$|u_1(x,t) - u_2(x,t)| < \epsilon \quad \forall (x,t)$$

Proof :- By d'Alembert's formula,

$$(u_1 - u_2)(x,t) = \frac{f_1(x+ct) - f_2(x+ct)}{2} + \frac{f_1(x-ct) - f_2(x-ct)}{2}$$
$$+ \frac{1}{2c}\int_{x-ct}^{x+ct} (g_1 - g_2)(s)\, ds$$
$$+ \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} \left(f_1(s,\tau) - f_2(s,\tau)\right) ds\, d\tau .$$

So, $$|u_1 - u_2| \le \delta + \frac{1}{\cancel{2c}}\, \cancel{2c}t\,\delta + \frac{1}{2\cancel{c}}\, \delta\, \cancel{c}\, t^2$$
$$\lesssim \delta(1 + T + T^2) < \epsilon$$

with $\delta < \dfrac{\epsilon}{1+T+T^2}$.

# Reflection of Waves :-

## [A] Semi-infinite string with a fixed End :

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \quad , \quad 0 < x < \infty \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \\ u(0,t) = 0 \quad \text{(Dirichlet BC)} \end{cases}$$

- Consider the odd extensions of $f$ and $g$:

$$\bar{f}(x) = \begin{cases} f(x) , & x > 0 \\ -f(-x) , & x < 0 \end{cases}$$

$$\bar{g}(x) = \begin{cases} g(x) , & x > 0 \\ -g(-x) , & x < 0 . \end{cases}$$

Then,

$$\begin{cases} \bar{u}_{tt} - c^2 \bar{u}_{xx} = 0 \quad , \quad -\infty < x < \infty \\ \bar{u}(x,0) = \bar{f}(x) \\ \bar{u}_t(x,0) = \bar{g}(x) \end{cases}$$

As $\bar{f}$ & $\bar{g}$ are odd fn., $\bar{u}$ is odd.

and $\bar{u}(-x,t) = -\bar{u}(x,t)$ , so that

$$\bar{u}(0,t) = 0 .$$

$$\bar{u}(x,t) = \frac{\bar{f}(x+ct) + \bar{f}(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \bar{g}(s)\, ds .$$

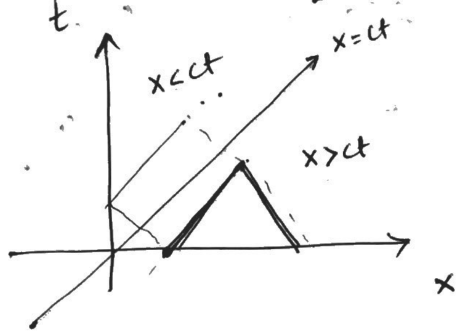
For $x \geq ct$.

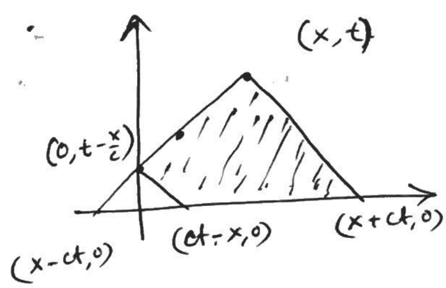
$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

for $x < ct$,

$$u(x,t) = \frac{f(x+ct) - f(ct-x)}{2} + \frac{1}{2c}\left[\int_{x-ct}^{0} -g(-s)\,ds + \int_{0}^{x+ct} g(s)\,ds\right]$$

$$= \frac{f(x+ct) - f(ct-x)}{2} + \frac{1}{2c}\int_{ct-x}^{x+ct} g(s)\,ds.$$




**B** Semi-infinite string with a free end :-

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & 0 < x < \infty \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \\ u_x(0,t) = 0 & (\text{Neumann BC}) \end{cases}$$

Consider the even extensions of $f$ and $g$:

$$\bar{f}(x) = \begin{cases} f(x), & x \geq 0 \\ f(-x), & x < 0 \end{cases} \quad \text{and } \bar{g}.$$

Then,

$$\begin{cases} \bar{u}_{tt} - c^2 \bar{u}_{xx} = 0 & -\infty < x < \infty \\ \bar{u}(x,0) = \bar{f}(x) \\ \bar{u}_t(x,0) = \bar{g}(x). \end{cases}$$

As $\bar{f}$ and $\bar{g}$ are even, $\bar{u}$ is even

$\bar{u}(-x,t) = \bar{u}(x,t)$ so that

$\bar{u}_x(0,t) = 0.$

$$\bar{u}(x,t) = \frac{\bar{f}(x+ct) + \bar{f}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \bar{g}(s)\,ds$$

For $x \geq ct$.

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

for $x < ct$

$$u(x,t) = \frac{f(x+ct) + f(ct-x)}{2} + \frac{1}{2c}\left\{\int_0^{ct-x} g(s)\,ds + \int_0^{x+ct} g(s)\,ds\right\}$$

## $\boxed{c}$ Bounded String :-

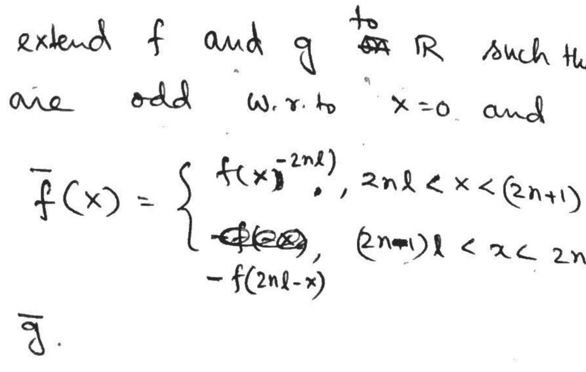
$$(*) \cdots \begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < l \\ u(x,0) = f(x) & \\ u_t(x,0) = g(x) & \\ u(0,t) = 0 = u(l,t) & , \; t \geq 0 \end{cases}$$

Method 1 (by reflection) (Strauss)

We extend $f$ and $g$ to $\mathbb{R}$ such that both $f$ and $g$ are odd w.r.to $x=0$ and $x=l$.

$$\bar{f}(x) = \begin{cases} f(x-2nl), & 2nl < x < (2n+1)l \\ -f(2nl-x), & (2n-1)l < x < 2nl \end{cases}, \quad n = 0, \pm 1, \pm 2, \ldots$$

Ill.ly for $\bar{g}$.



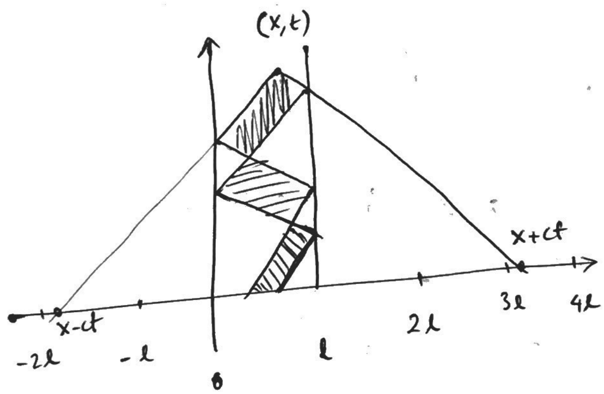
$\bar{f}$ and $\bar{g}$ are of period $2l$. i.e.

$$\bar{f}(x+2l) = \bar{f}(x), \quad \bar{g}(x+2l) = \bar{g}(x). \quad \forall x.$$

Then,
$$\begin{cases} \bar{u}_{tt} - c^2 u_{xx} = 0 & -\infty < x < \infty \\ \bar{u}(x,0) = \bar{f}(x) \\ \bar{u}_t(x,0) = \bar{g}(x) \end{cases}$$

and $\bar{u}\big|_{(0,l)} = u$.

$$\bar{u}(x,t) = \frac{\bar{f}(x+ct) + \bar{f}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \bar{g}(s)\,ds, \quad x \in \mathbb{R}.$$




$-2l < x-ct < -l$

$3l < x+ct < 4l$

$$u(x,t) = \frac{f(x-ct+2l) - f(4l-x-ct)}{2} + \frac{1}{2c}\int_{x-ct+2l}^{4l-x-ct} g(s)\,ds.$$

## Method 2 (Seperation of variable / Fourier Method)

Let, $u(x,t) = F(x)\,G(t)$. , $x \in (0,l)$, $t \geq 0$ ... (I)

$u(0,t) = 0 = u(l,t) \Rightarrow$

$$F(0)\,G(t) = 0 = F(l)\,G(t).$$

$$\Rightarrow F(0) = 0 = F(l). \quad \text{--- (II)}$$

From the eqn.:

$$F(x)\,G''(t) = c^2 F''(x)\,G(t)$$

$$\Rightarrow \frac{F''(x)}{F(x)} = \frac{G''(t)}{c^2 G(t)} = -\lambda, \text{ say.}$$

because, each group is a fn. of $x/t$ only.

[At the end, we will show $\lambda > 0$].

$$\text{So,} \quad \left.\begin{aligned} F'' + \lambda F &= 0 \\ F(0) = F(l) &= 0 \end{aligned}\right\} \text{ and } G'' + \lambda c^2 G = 0$$

These are ~~Sturm~~ Sturm-Liouville problems. The values of $\lambda$ for which it has non-trivial soln., are called eigenvalues, and corresponding soln. are called eigenfns.

<u>Case-I</u> $\lambda = -\mu^2 < 0$. The gen. soln. of

$$F'' - \mu^2 F = 0 \text{ is}$$

$$F(x) = A e^{\mu x} + B e^{-\mu x}$$

Now, $F(0) = 0 = F(l) \Rightarrow A = 0 = B$.

So, there are no - nontrivial soln. in this case.

<u>Case-II</u> $\lambda = 0$ : No nontrivial soln.

Case-III $\lambda = \mu^2 > 0$. The general sol$\underline{n}$ is:

$$F(x) = A \sin \mu x + B \cos \mu x.$$

$$F(0)=0 \Rightarrow B = 0$$

$$F(l)=0 \Rightarrow A \sin(\mu l) = 0$$

$$\Rightarrow \sin(\mu l) = 0, \quad A \neq 0$$

So, $\mu_n = \dfrac{n\pi}{l}$, $n = 1, 2, \ldots$

So, the eigenvalues are:

$$\lambda_n = \mu_n^2 = \left(\frac{n\pi}{l}\right)^2, \quad n = 1, 2, \ldots$$

and the eigen-fn$\underline{s}$ are:

$$F_n(x) = \sin\left(\frac{n\pi x}{l}\right), \quad n = 1, 2, \ldots$$

Solving the other eq$\underline{n}$:

$$G_n(t) = K_n \sin\left(\frac{cn\pi t}{l}\right) + D_n \cos\left(\frac{cn\pi t}{l}\right).$$

Therefore, there are an infinite no of seperated sol$\underline{n}$ of $(*)$. So the sol$\underline{n}$ is:

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ K_n \sin\left(\frac{cn\pi t}{l}\right) + D_n \cos\left(\frac{cn\pi t}{l}\right) \right\} \sin\left(\frac{n\pi x}{l}\right)$$

provided it converges. $K_n$ and $D_n$ are choosen such that the ICs are satisfied. (Superposition principle).

Now, $u(x,0) = f(x)$, $u_t(x,0) = g(x)$.

So, $\displaystyle\sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{l}\right) = f(x)$ and

$$\sum_{n=1}^{\infty} \frac{\pi n c}{l} K_n \sin\left(\frac{n\pi x}{l}\right) = g(x).$$

The eq$\underline{n}$ will be satisfied if $f(x)$ & $g(x)$ are represented by Fourier Sine series.

$$\therefore \quad D_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx.$$

$$\text{and} \quad K_n = \frac{2}{l} \cdot \frac{l}{n\pi c} \int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx.$$

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Solve:

$$\begin{cases} u_{tt} - c^2 u_{xx} = \phi(x,t), & 0<x<l, \quad t \geq 0 \\ u(x,0) = f(x), & u_t(x,0) = g(x) \\ u(0,t) = p(t), & u(l,t) = q(t) \end{cases}$$

Let, $U(x,t) = \frac{1}{l}\left[(l-x)\,p(t) + x\,q(t)\right]$

and $v(x,t) = (u - U)(x,t)$.

Then, $v_{tt} - c^2 v_{xx} = \phi(x,t) - \frac{1}{l}\left[(l-x)\,p''(t) + x\,q''(t)\right]$

$= F(x,t)$, say.

$v(x,0) = f(x) - U(x,0) = \psi(x)$

$v_t(x,0) = g(x) - U_t(x,0) = G(x)$.

$v(0,t) = p(t) - p(t) = 0$, $\quad v(l,t) = 0$.

So, enough to solve:

$$\begin{cases} v_{tt} - c^2 v_{xx} = F(x,t), & 0<x<l. \\ v(x,0) = \psi(x), & v_t(x,0) = G(x) \\ v(0,t) = 0, & v(l,t) = 0 \end{cases}$$

Solve it by Duhamel.

## Energy :-

$R = \{(x,t) : 0 < x < l,\ 0 < t < \infty\}$. Let, $u \in C^2(R)$ is a soln of

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \\ u(0,t) = 0 = u(l,t) \end{cases}$$

$KE(t) = \frac{1}{2}\int_0^l u_t^2(x,t)\,dx$ is called the kinetic energy.

$PE(t) = \frac{1}{2}\int_0^l c^2 u_x^2(x,t)\,dx$ is called the potential "

$E(t) = KE(t) + PE(t) = \frac{1}{2}\int_0^l (u_t^2 + c^2 u_x^2)\,dx$ is called the total energy of the system at time $t$.

**Theorem 11.** If $u \in C^2(R)$ is a soln, then, $E(t)$ is constant, $E(t) = E(0)$.

$$\begin{aligned}
\frac{dE}{dt} &= \frac{d}{dt}\frac{1}{2}\int_0^l (u_t^2 + c^2 u_x^2)\,dx \\
&= \frac{1}{2}\int_0^l (2u_t u_{tt} + 2c^2 u_x u_{xt})\,dx \\
&= \int_0^l u_t u_{tt}\,dx + c^2\int_0^l u_x u_{xt}\,dx \\
&= \int_0^l u_t u_{tt}\,dx + c^2 \overset{0}{\cancel{\left[u_x u_t\right]_0^l}} - c^2\int_0^l u_{xx} u_t\,dx \\
&= \int_0^l u_t (u_{tt} - c^2 u_{xx})\,dx = 0.
\end{aligned}$$

So, $E(t) = K = E(0)$.

Thm 12 The smooth soln of

$$(*) \cdots \begin{cases} u_{tt} - c^2 u_{xx} = \phi(x,t), & 0<x<l \\ u(x,0) = f(x), & u_t(x,0) = g(x) \\ u(0,t) = h(t), & u(l,t) = \psi(t) \end{cases}$$

is unique.

Proof: If possible let, $u_1$ & $u_2$ satisfy $(*)$.

Then, $v = u_1 - u_2$ satisfies

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 \\ v(x,0) = 0 = v_t(x,0) \\ v(0,t) = 0 = v(l,t) \end{cases}$$

If $E(t) = \frac{1}{2}\int_0^l (v_t^2 + c^2 v_x^2)\,dx$, then $E(t) = E(0)$

But, $E(0) = \frac{1}{2}\int_0^l (v_t^2 + c^2 v_x^2)\Big|_{t=0} = 0$.

So, $\frac{1}{2}\int_0^l (v_t^2 + c^2 v_x^2)\,dx = 0 \quad \forall \ t>0$

i.e. $v_t(x,t) = 0 = v_x(x,t)$.

So, $v(x,t) = \text{constant}$. But, $v(x,0) = 0$.

So, $v(x,t) = 0 \Rightarrow u_1 = u_2$. The soln is unique

H.W

$$\begin{cases} u_{tt} - u_{xx} + u_t = 0, & 0<x<l,\ t>0 \\ u(x,0) = \phi(x), & u_t(x,0) = \psi(x) \\ u(0,t) = 0 = u(l,t) \end{cases}$$

Prove that the energy fn is a decreasing fn