Heat Equation :-

$$\begin{cases} u_t - Ku_{xx} = f(x,t), & x \in \mathbb{R}, t > 0 \\ u(x,0) = \phi(x) \end{cases} \quad \dots (1)$$

K: heat conductivity.

In $\mathbb{R}^n$:

$$\begin{cases} u_t - K\Delta u = f(x,t), & \text{in } \mathbb{R}^n, t>0 \\ u(\underline{x},0) = \phi(\underline{x}) & \underline{x} \in \mathbb{R}^n \\ |u| < \infty \end{cases}$$

Fundamental Solution :-

The eqn

$$\begin{cases} u_t - Ku_{xx} = 0, & x \in \mathbb{R}, t > 0 \\ u(x,0) = \phi(x) \end{cases} \quad \dots (11)$$

Satisfies the following invariance properties:

a) if $u(x,t)$ is a soln, $u(x-y,t)$ is also a soln for a fixed y.

b) Any derivative $(u_x, u_t, u_{xx} \dots)$ of a soln is again a soln.

c) A linear combination of solns is again a soln.

d) An integral of soln is again a soln. If $u(x,t)$ is a soln, so is $u(x-y,t)$ and so is

$$v(x,t) = \int_{-\infty}^{\infty} u(x-y,t)\, g(y)\, dy.$$

E(0).

unique.

for any $f^{\underline{n}}$ $g(y)$. (Assume convergence)

e) $v(x,t) = u(\sqrt{a}\,x, at)$, $a>0$ is also a $soln$.

$$v_t = a\,u_t(\sqrt{a}\,x, at)$$

$$v_x = \sqrt{a}\,u_x(\sqrt{a}\,x, at)$$

$$v_{xx} = a\,u_{xx}(\sqrt{a}\,x, at)$$

$$\Rightarrow \quad v_t - K v_{xx} = 0.$$

Let, $z = \sqrt{a}\,x$, $\tau = at$, $a>0$.

$$\frac{z^2}{K\tau} = \frac{x^2}{Kt}$$ and the heat $eq^{\underline{n}}$ is invariant under $(x,t) \to (z,\tau)$.

Let, us look for a $soln$ of heat $eq^{\underline{n}}$ having the form

$$Q(x,t) = w\left(\frac{x}{\sqrt{4Kt}}\right)$$

Substituting this expression in the $eq^{\underline{n}}$, we get:

$$Q_t = w'\left(\frac{x}{\sqrt{4Kt}}\right) \cdot \frac{x}{\sqrt{4K}} \cdot \left(-\frac{1}{2}\right) t^{-3/2}$$

$$Q_{xx} = w''\left(\frac{x}{\sqrt{4Kt}}\right) \frac{1}{4Kt}$$

Let, $p = \frac{x}{\sqrt{4Kt}}$ and $Q_t - KQ_{xx} = 0 \Rightarrow$

$$\frac{1}{\sqrt{4Kt}}\left(\frac{x}{2t}\,w' + w''\,\frac{K}{\sqrt{4Kt}}\right) = 0$$

$$\Rightarrow \quad \omega'' + \frac{x}{\sqrt{kt}}\,\omega' = 0$$

$$\therefore \quad \omega'' + 2p\,\omega' = 0$$

Hence, $\quad \omega'(p) = K_1 e^{-p^2}$

$$\therefore \quad \omega(p) = K_1 \int_0^p e^{-s^2}\,ds + K_2$$

$$\therefore \quad Q(x,t) = K_1 \int_0^{x/\sqrt{4kt}} e^{-s^2}\,ds + K_2$$

We choose an ~~ind~~ initial condn.

$$Q(x,0) = \begin{cases} 1, & x > 0 \\ 0, & x < 0. \end{cases}$$

then, $Q(x,0) = \lim_{t \to 0+} Q(x,t) = \begin{cases} K_1 \frac{\sqrt{\pi}}{2} + K_2, & x > 0 \\ -K_1 \frac{\sqrt{\pi}}{2} + K_2, & x < 0 \end{cases}$

$$\therefore \quad K_2 = 1/2, \quad K_1 = 1/\sqrt{\pi}.$$

So, $$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-s^2}\,ds$$

By (b), $Q_x(x,t)$ is also a soln., let, $K(x,t) = Q_x(x,t)$.

i.e. $$K(x,t) = \frac{1}{\sqrt{4\pi kt}}\, e^{-\frac{x^2}{4kt}}, \quad t > 0$$

By (d), for any $\phi(x)$,

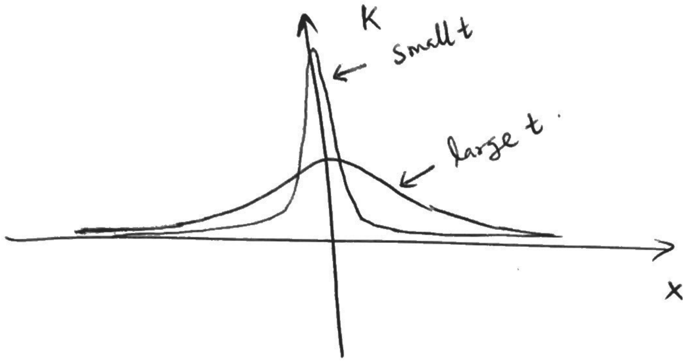
$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\,\phi(y)\,dy$$

is also a soln. of (11). (Poisson Formula)

Property:

$$\int_{-\infty}^{\infty} K(x,t)\,dx = 1 \quad (\text{for each } t>0)$$

This $K(x,t)$ is called the <u>fundamental solⁿ</u> or the <u>diffusion kernel</u>. .



Theorem:- Let, $\Phi : \mathbb{R} \to \mathbb{R}$ be a cont. & bounded fⁿ

and $u(x,t) = (K * \Phi)(x,t)$.

Then,

1) $u \in C^\infty(\mathbb{R} \times (0,\infty))$

11) u solves $u_t - k u_{xx} = 0$, $t>0$

111) $\lim_{\substack{t\to 0+ \\ (x,t)\to(x_0,0)}} u(x,t) = \Phi(x_0)$, $x_0 \in \mathbb{R}$.

Proof:- (111). ϕ is cont. For $\epsilon>0$, choose $\delta>0$ s.t.

$|\Phi(y) - \Phi(x_0)| < \frac{\epsilon}{2}$. whenever $y \in B_\delta(x_0)$.

~~$|u(x,t) - \Phi(x_0)|$~~

If $x \in B_{\delta/2}(x_0)$, then,

$$|u(x,t) - \Phi(x_0)| = \left| \int_{\mathbb{R}} K(x-y,t)\,\Phi(y)\,dy - \int_{\mathbb{R}} K(x_0-y,t)\,\Phi(x_0)\,dy \right|$$

$$= \left| \int_{\mathbb{R}} K(x-y,t)\,(\Phi(y) - \Phi(x_0))\,dy \right|$$

$$\leq \int_{B_\delta(x_0)} K(x-y,t)\,|\phi(y)-\phi(x_0)|\,dy$$

$$+ \int_{|y-x_0|\geq\delta} K(x-y,t)\,|\phi(y)-\phi(x_0)|\,dy$$

$$\leq \frac{\epsilon}{2} + J.$$

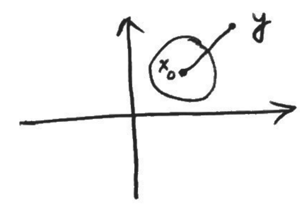
$$J = \int_{|y-x_0|\geq\delta} K(x-y,t)\,|\phi(y)-\phi(x_0)|\,dy$$

$$|y-x_0| \leq |y-x|+|x-x_0| \leq |y-x| + \delta/2$$

$$\leq |y-x| + \tfrac{1}{2}|y-x_0|$$

$$\Rightarrow \quad \underline{|y-x| \geq \tfrac{1}{2}|y-x_0|}.$$

Now, $$J \leq \frac{2\|\phi\|_\infty}{\sqrt{4\pi k t}} \int_{|y-x_0|\geq\delta} e^{-\frac{(x-y)^2}{4kt}}\,dy$$



$$\leq \frac{C}{\sqrt{t}} \int_{|y-x_0|\geq\delta} e^{-\frac{(y-x_0)^2}{16kt}}\,dy$$

$$= C_1 \int_{\delta/\sqrt{16kt}}^{\infty} e^{-s^2}\,ds \qquad \frac{(y-x_0)}{\sqrt{16kt}} = s, \quad \frac{dy}{\sqrt{16kt}} = ds$$

$$\to 0 \text{ as } t \to 0+.$$

So, for sufficiently small t, $J < \epsilon/2$.

i.e. $|u(x,t)-\phi(x_0)| < \epsilon$.

The Duhamel principle :-

$$(I) \ldots \begin{cases} u_t - k u_{xx} = f(x,t), & x \in \mathbb{R},\ t>0 \\ u(x,0) = \phi(x) \end{cases}$$

Consider

$$(II) \ldots \begin{cases} v_t - k v_{xx} = 0 & x \in \mathbb{R},\ t>0 \\ v(x,0) = \phi(x) \end{cases}$$

and

$$(III) \ldots \begin{cases} w_t - k w_{xx} = f(x,t), & x \in \mathbb{R},\ t>0 \\ w(x,0) = 0 \end{cases}$$

then, $u = v + w$ solves (I). and.

$$v(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\, dy .$$

Need to solve (III).

Consider $v(x,t;\tau)$ that satisfies:

$$\begin{cases} v_t - k v_{xx} = 0, & x \in \mathbb{R},\ t > \tau \\ v(x,\tau;\tau) = f(x,\tau) \end{cases}$$

for each $\tau \in (0,\infty)$.

If $\bar{v}(x,t-\tau) = v(x,t;\tau)$, then,

$$\begin{cases} \bar{v}_s - k \bar{v}_{xx} = 0, & s>0,\ x \in \mathbb{R}. \\ \bar{v}(x,0) = f(x,\tau) \end{cases}$$

So, $\bar{v}(x,s) = v(x,t;\tau)$

$$= \frac{1}{\sqrt{4\pi k(t-\tau)}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4k(t-\tau)}} f(y,\tau)\, dy.$$

Theorem:-

$$w(x,t) = \int_0^t v(x,t;\tau)\, d\tau \quad \text{solves (III).}$$

Proof is similar to wave eqn.

So, the complete soln. for (I) is:

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\, dy$$

$$+ \int_0^t \frac{1}{\sqrt{4\pi(t-\tau)}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4k(t-\tau)}} f(y;\tau)\, dy\, d\tau.$$

Half Line:-

Dirichlet problem

$$\begin{cases} u_t - k u_{xx} = f(x,t), & 0 < x < \infty,\ t > 0 \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \end{cases}$$

Let, $U(x,t) = u(x,t) - g(t)$ then, U satisfies:

$$\begin{cases} U_t - k U_{xx} = f(x,t) - g'(t) \\ U(x,0) = \phi(x) - g(0) \\ U(0,t) = 0 \end{cases}$$

Now use odd extension as in wave eqn.

Neumann problem :-

$$\begin{cases} w_t - K w_{xx} = f(x,t), & 0<x<\infty,\ t>0 \\ w(x,0) = \phi(x) \\ w_x(0,t) = h(t) \end{cases}$$

Let, $W(x,t) = w(x,t) - x\,h(t)$, then, W satisfies

$$\begin{cases} W_t - K W_{xx} = f(x,t) - x\,h'(t) \\ W(x,0) = \phi(x) - x\,h(0) \\ W_x(0,t) = 0 \end{cases}$$

Use even extension as in the wave eqn.

Heat Conduction on finite Rod :-

$$\begin{cases} u_t - K u_{xx} = 0, & 0<x<l,\ t>0 \\ u(x,0) = \phi(x) \\ u(0,t) = 0 = u(l,t) \end{cases}$$

Homogeneous PDE, BCs. $\Rightarrow$ Séperation of variable.

Assume $u(x,t) = F(x)\,G(t)$.

then,

$$FG' - K F'' G = 0$$

$$\Rightarrow \frac{G'}{KG} = \frac{F''}{F} = -\mu^2 = \lambda, \quad \mu > 0$$

with $F(0) = 0 = F(l)$. The only non-trivial soln occurs when, $\lambda = -\mu^2$.

Now, the eigen-value problem:

$$\begin{cases} F'' + \mu^2 F = 0 \\ F(0) = F(l) = 0 \end{cases}$$

has soln:

$$F(x) = A\cos(\mu x) + B\sin(\mu x)$$

$$F(0) = 0 \Rightarrow A = 0$$

$$F(l) = 0 \therefore \Rightarrow B\sin(\mu l) = 0$$

$$B \neq 0 : \quad \sin(\mu l) = 0$$

$$\therefore \mu l = n\pi.$$

$$\therefore \mu_n = \frac{n\pi}{l}, \quad n = 1, 2, \ldots$$

These are the eigen-values. and the eigen-fns are:

$$F_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right), \quad n = 1, 2, \ldots$$

Also,

$$G' + \mu^2 k G = 0$$

$$\therefore G_n(t) = C_n e^{-\mu_n^2 kt}, \quad n = 1, 2, \ldots$$

$$= C_n e^{-\left(\frac{n\pi}{l}\right)^2 kt}$$

Hence, the non-trivial soln of the heat eqn which satisfies the two BCs. is:

$$u_n(x,t) = F_n(x)\, G_n(t)$$

$$= a_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\left(\frac{n\pi x}{l}\right), \quad n = 1, 2, \ldots$$

By superposition principle, the series soln is:

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\left(\frac{n\pi x}{l}\right).$$

and it should satisfy the IC: $u(x,0) = \Phi(x)$.

So, $\Phi(x)$ is represented by a Fourier Sine-Series with co-eff.

$$a_n = \frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

H.W :- Let, $\phi(x) = x(l-x)$

then, write the soln.

$$a_n = \frac{8l^2}{n^3\pi^3}. \quad n = 1, 3, 5, \ldots$$

$$\therefore \ u(x,t) = \frac{8l^2}{\pi^3}\sum_{n=1,3,5}^{\infty} \frac{1}{n^3} e^{-\left(\frac{n\pi}{l}\right)^2 Kt} \sin\left(\frac{n\pi x}{l}\right).$$

H.W: Instead of Dirichlet BCs, we have neumann Bc.

$$u_x(0,t) = 0 = u_x(l,t).$$

~~[illegible]~~

$$\mu_n = \left(\frac{n\pi}{l}\right)^2, \quad n = 0, 1 \ldots$$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi}{l}\right)^2 Kt} \cos\left(\frac{n\pi x}{l}\right).$$

with $\phi(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$: Fourier Cosine Series

$$a_n = \frac{2}{l}\int_0^l \phi(x)\cos\left(\frac{n\pi x}{l}\right)dx \quad \text{(Check)}.$$

Uniqueness by Energy Method :-

$$(*) \ldots \begin{cases} u_t - K u_{xx} = f(x,t), & 0<x<l,\ t>0 \\ u(x,0) = \phi(x) \\ u(0,t) = g_1(t),\ u(l,t) = g_2(t) \end{cases}$$

Defn :- The energy integral is given by:

$$E(t) = \frac{1}{2K}\int_0^l u^2\,dx.$$

Thm 13. Let, $u \in C^1$ be a soln of $(*)$. The soln is unique.

Proof :- If possible, let, $u_1(x,t)$ and $u_2(x,t)$ are two soln of $(*)$.

Let, $v(x,t) = (u_1 - u_2)(x,t)$

Then,
$$\begin{cases} v_t - K v_{xx} = 0, & 0<x<l,\ t>0 \\ v(x,0) = 0 \\ v(0,t) = 0 = v(l,t) \end{cases}$$

$$\frac{dE}{dt} = \frac{d}{dt}\left\{\frac{1}{2K}\int_0^l v^2(x,t)\,dx\right\}$$

$$= \frac{1}{K}\int_0^l v v_t\,dx$$

$$= \int_0^l v v_{xx}\,dx$$

$$= [v v_x]_0^l - \int_0^l v_x^2\,dx$$

$$= v(l,t)\,v_x(l,t) - v(0,t)\,v_x(0,t) - \int_0^l v_x^2\,dx$$

$$= -\int_0^l v_x^2\,dx \leq 0$$

Also, $E(0) = \frac{1}{2k}\int_0^l v^2(x,0)\,dx = 0.$

So, $E(t)$ is a decreasing fn and $E(t) \leq E(0) = 0$

By defn, $E(t) \geq 0. \quad \forall t$

Hence, $E(t) = 0 \implies v(x,t) = 0. \quad \forall (x,t)$

So, $u_1 = u_2$ and the soln is unique.

Inhomogeneous eqn :-

$$(1) \cdots \begin{cases} u_t - k u_{xx} = f(x,t), & 0<x<l,\ t>0 \\ u(x,0) = \phi(x) \\ u(0,t) = 0 = u(l,t) \end{cases}$$

Suppose a soln of (1) looks like:

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t)\, e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\left(\frac{n\pi x}{l}\right).$$

Substituting in (1), we get:

$$\sum_{n=1}^{\infty} A_n'(t)\, e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\left(\frac{n\pi x}{l}\right) = \sum_{n=1} f_n(t) \sin\left(\frac{n\pi x}{l}\right)$$

where $f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi x}{l}\right).$

& $f_n(t) = \frac{2}{l}\int_0^l f(x,t) \sin\left(\frac{n\pi x}{l}\right) dx.$

Equating the co-eff. we get:

$$A_n'(t) = f_n(t).\, e^{\left(\frac{n\pi}{l}\right)^2 kt}.$$

$$\therefore A_n(t) = A_n(0) + \int_0^t e^{\left(\frac{n\pi}{l}\right)^2 ks} f_n(s)\,ds$$

$$\phi(x) = u(x,0) = \sum_{n=1}^{\infty} A_n(0) \sin\left(\frac{n\pi x}{l}\right)$$

$$\therefore \quad A_n(0) = \frac{2}{l}\int_0^l \phi(x) \sin\left(\frac{n\pi x}{l}\right) dx.$$

Therefore the soln. of (1) is :

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} + \int_0^t e^{\left(\frac{n\pi}{l}\right)^2 k(s-t)} f_n(s)\, ds \right) \sin\left(\frac{n\pi x}{l}\right)$$

where
$$f_n(s) = \frac{2}{l}\int_0^l f(x,s) \sin\left(\frac{n\pi x}{l}\right) dx$$
$$a_n = \frac{2}{l}\int_0^l \phi(x) \sin\left(\frac{n\pi x}{l}\right) dx\,, \quad n = 1, 2, \ldots$$

Non-homogeneous BCs :-

$$\begin{cases} u_t - k u_{xx} = 0, & 0 < x < l,\ t > 0 \\ u(x,0) = \phi(x) \\ u(0,t) = g(t), \quad u(l,t) = h(t) \end{cases}$$

Let, $U(x,t) = \frac{1}{l}\left[(l-x)\, g(t) + x\, h(t)\right]$

Then, $w(x,t) = (u - U)(x,t)$ satisfies:

$$\begin{cases} w_t - k w_{xx} = -\frac{1}{l}\left\{(l-x)\, g'(t) + x\, h'(t)\right\} \\ w(x,0) = \phi(x) - \frac{1}{l}\left\{(l-x)\, g(0) + x\, h(0)\right\} \\ w(0,t) = w(l,t) = 0 \end{cases}$$

This can be solved by previous technique.

Thm 14 (Maximum - Minimum principle)

Let, $R = \{(x,t) : 0 \le x \le l;\ 0 \le t \le T\}$

$\Gamma = \{(x,t) \in R : t = 0 \text{ or } x = 0 \text{ or } x = l\}$

Let, $u(x,t)$ be continuous fn. in R which satisfies $u_t - k u_{xx} = 0$ in $R - \Gamma$. Then,

$$\max_R u(x,t) = \max_\Gamma u(x,t)$$

and

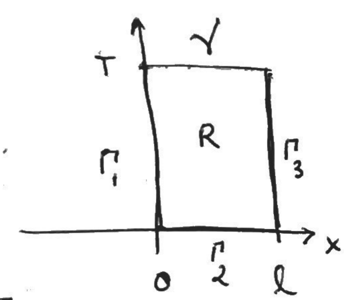
$$\min_R u(x,t) = \min_\Gamma u(x,t).$$

Proof :- Let, $M = \max_\Gamma u(x,t)$.

To show $\max_R u(x,t) \le M$. $(\Gamma \subseteq R)$

Consider $v(x,t) = u(x,t) + \epsilon x^2$, $\epsilon > 0$.

Then for $(x,t) \in R - \Gamma$



$$v_t - k v_{xx} = u_t - k u_{xx} - 2\epsilon k$$

$$= -2\epsilon k < 0$$

and $v(x,t) \le M + \epsilon l^2$, $(x,t) \in \Gamma$.

If $v(x,t)$ attains its maximum at an interior pt (x_1, t_1), then,

$$v_t - k v_{xx}\Big|_{(x_1,t_1)} \geq 0, \text{ a contradiction.}$$

So, v attains max at a pt. of

$$\partial R = \Gamma \cup \gamma, \quad \gamma = \{(x,t) \in R : t = T\}$$

Suppose $v(x,t)$ has a maximum at $(\bar{x},T)$, $0<\bar{x}<l$.

Then, $v_x(\bar{x},T)=0$, $v_{xx}(\bar{x},T)\leq 0$

Also, $v(\bar{x},T)\geq v(\bar{x},T-\delta)$, $0<\delta<T$.

Then, $$v_t(\bar{x},T)=\lim_{\delta\to 0+}\frac{v(\bar{x},T)-v(\bar{x},T-\delta)}{\delta}$$

$$\geq 0$$

Hence, $v_t - k v_{xx}\Big|_{(\bar{x},T)} \geq 0$, a contradiction.

Therefore, $M_1 = \max_R v(x,t) = \max_\Gamma v(x,t)$
$\leq M+\epsilon l^2$

So, $u(x,t)\leq M+\epsilon(l^2-x^2)$ on R.

Letting $\epsilon\to 0$, we get, $u(x,t)\leq M$ on R.

So, $\max_R u(x,t) = \max_\Gamma u(x,t)$

For minimum, take $w(x,t) = -u(x,t)$.

- <u>Uniqueness</u> :- If u_1 & u_2 are two solns of

$$\begin{cases} u_t - k u_{xx} = f(x,t), & 0<x<l,\ t>0 \\ u(x,0) = \phi(x) \\ u(0,t) = g(t), \ u(l,t) = h(t) \end{cases}$$

then $v=u_1-u_2$ satisfies:

$$\begin{cases} v_t - k v_{xx} = 0 \\ v(x,0) = 0 \\ v(0,t) = 0 = v(l,t) \end{cases}$$

So, $\max_R v(x,t) = \min_R v(x,t) = 0. \Rightarrow v=0$
$\Rightarrow u_1 = u_2$

Tikhonov Example :-

$$g(t) = \begin{cases} e^{-1/t^2}, & t>0 \\ 0, & t \le 0 \end{cases}$$

$$u_t - k u_{xx} = 0, \quad u(x,0) = 0 \quad \cdots (*)$$

$$u(x,t) = \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}$$

satisfies (*). But, $v(x,t) \equiv 0$ is also a solⁿ.

Hence, the Cauchy problem (*) does not have a unique solⁿ unless further restrictions on $u(x,t)$.

i.e. $|u(x,t)| \le M e^{ax^2}, \; t \ge 0$

In fact, there are infinitely many solⁿ to (*)

$$g_\alpha(t) = \begin{cases} e^{-t^{-\alpha}}, & t>0 \\ 0, & t \le 0 \end{cases}, \quad \alpha > 1$$

H.W :- Let, $u \in C^2$. Solves:

$$\begin{cases} u_t - k u_{xx} = 0, & 0<x<l, \; t>0 \\ u(x,t) = 0, & x = 0, l \\ u(x,0) = 0 \end{cases}$$

If $u(x,T) = 0$, then $u \equiv 0$.

[Hint : Use energy method (Heat2)]

Strong Max. Principle :- If Ω (interior) $= R - \partial R$ is connected and $\exists$ $(x_0,t_0) \in R - \Gamma$ s.t. $u(x_0,t_0) = \max_R u(x,t)$, then, u is constant $\mathring{R}_{t_0}$ (Evans) $= \{(x,t) \mid 0<x<l, \; 0<t\le t_0\}$

(Version 2) Maximum Principle (weak form) :-

ω = open bdd set in $\mathbb{R}^n$

$$\Omega = \{ (\underline{x},t) \in \mathbb{R}^{n+1} : \underline{x} \in \omega, \quad 0<t<T \}$$

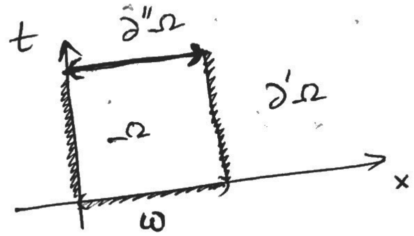
$$\partial''\Omega = \{ (\underline{x},t) : \underline{x} \in \omega, \; t = T \}$$

$$\partial'\Omega = \{ (\underline{x},t) : \underline{x} \in \partial\omega, \; 0 \le t \le T \} \cup \{ (\underline{x},t) : \underline{x} \in \omega, t=0 \}$$

$$\partial\Omega = \partial\Omega' \cup \partial\Omega''.$$

If $u_t - \Delta u \le 0$ in Ω, then,

$$\max_{(\underline{x},t) \in \bar{\Omega}} u(\underline{x},t) = \max_{\partial'\Omega} u(\underline{x},t).$$



Stability :-

Let, u_1 and u_2 be two solns of

$$\begin{cases} u_t - K u_{xx} = F(x,t) & x \in \Omega \\ u_i(x,0) = \Phi_i(x) \\ u_i(x,t) = h_i(t), & x \in \partial\Omega \end{cases}$$

Let, $\epsilon = \max_{\Omega} |\phi_1 - \phi_2| + \max_{\partial\Omega \times \{t>0\}} |h_1 - h_2|$, then

$$|u_1 - u_2| \le \epsilon, \quad (x,t) \in \Omega \times [0,T]$$

Problems :-

1. The max principle is not valid for parabolic eqn 6.
with variable co-eff. Verify for $u_t - x u_{xx} = 0$ in
$R = \{ (x,t) : -2 \le x \le 2,\ 0 \le t \le 1 \}$ has a soln
$u(x,t) = -2xt - x^2$ and $\max\limits_{R} u(x,t) = u(-1,1) = 1$.

2.
$$\begin{cases} u_t - u_{xx} = 0, & 0 < x < 2,\ t > 0 \\ u(x,0) = x(2-x) & 0 \le x \le 2 \\ u(0,t) = 0 = u(2,t) \end{cases}$$ 7.

S.t. i) $0 < u(x,t) < 1 \quad \forall\, t > 0,\ 0 < x < 2$

ii) $u(x,t) = u(2-x,t)$ for every $t \geq 0$, $0 \le x \le 2$

iii) $\displaystyle\int_0^2 u^2(x,t)\,dx \le \frac{16}{15} \quad \forall\, t \geq 0 \quad \left(E(t) \le E(0) \right)$

3. ~~Proof from first principles~~

Prove that $Q(x,t) = Q(\sqrt{a}\,x, at)$ satisfies the heat eqn. for $a > 0$. Choose $a = \frac{1}{4kt}$ to prove it has a special form $Q(x,t) = q\left(\frac{x}{\sqrt{4kt}}\right)$.

4. Prove the uniqueness of the soln of the diffusion problem:

$$u_t - k u_{xx} = f(x,t), \quad 0 < x < l,\ t > 0$$
$$u(x,0) = \phi(x)$$
$$u_x(0,t) = g(t), \quad u_x(l,t) = h(t)$$

by energy method.

5. Solve the eqn $u_t - k u_{xx} + bu = 0$, $-\infty < x < \infty$, $u(x,0) = \phi(x)$, $b > 0$. (Hint: $v(x,t) = e^{bt} u(x,t)$).

6. Prove that, wave eqn doesn't follow maximum principle.

[$u(x,t) = \sin x \sin t$ satisfies $u_{tt} - u_{xx} = 0$ in $[0,\pi]\times[0,\pi]$

with $u(x,0) = 0$, $u_t(x,0) = \sin x$, $u(0,t) = 0 = u(\pi,t)$.

$\max_R u(x,t) = 1 = u(\pi/2, \pi/2)$ in an interior pt.

But $u = 0$ along the boundary.].

7. <u>Comparison principle</u> :- If u & v ~~are~~ are two solns of $u_t - k u_{xx} = 0$ and $u \le v$ for $t = 0$, $x = 0$ and $x = l$

then $u \le v$ for $0 \le t < \infty$, $0 \le x \le l$.

8. Let, $Q = \{(x,t) \mid 0 < x < \pi,\ 0 < t \le T\}$. Let u

Solves

$$u_t - u_{xx} = 0 \quad \text{in } Q$$

$$u(0,t) = 0 = u(\pi,t), \quad 0 \le t \le T.$$

$$u(x,0) = \sin^2(x), \quad 0 \le x \le \pi.$$

Show that, $0 \le u(x,t) \le e^{-t}\sin(x)$ in Q.

[$v(x,t) = e^{-t}\sin(x) \Rightarrow$
$$\left.\begin{aligned} v_t - v_{xx} &= 0 \\ v(0,t) &= 0 = v(\pi,t) \\ v(x,0) &= \sin x. \end{aligned}\right\}$$

and $\sin x \ge \sin^2 x$ in $0 \le x \le \pi$.

So, $u(x,t) \le v(x,t)$ in Q by Max principle

Also, by Min-principle,

<u>$0 \le u(x,t)$</u> $\le v(x,t)$ in Q.]